



DEPARTMENT OF EDUCATION

GRADE 12 MATHEMATICS A

12.2: TRIGONOMETRY AND VECTORS



FODE DISTANCE LEARNING



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PAPUA NEW GUINEA



GRADE 12

MATHEMATICS A

UNIT MODULE 2

TRIGONOMETRY AND VECTORS

TOPIC 1: TRIGONOMETRY

TOPIC 2: TRIGONOMETRIC IDENTITIES

TOPIC 3: VECTORS



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MR. DEMAS TONGOGO
Principal-FODE



Flexible Open and Distance Education
Papua New Guinea

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SECRETARY'S MESSAGE

Achieving a better future by individuals students, their families, communities or the nation as a whole, depends on the curriculum and the way it is delivered.

This course is part and parcel of the new reformed curriculum – the Outcome Base Education (OBE). Its learning outcomes are student centred and written in terms that allow them to be demonstrated, assessed and measured.

It maintains the rationale, goals, aims and principles of the National OBE Curriculum and identifies the knowledge, skills, attitudes and values that students should achieve.

This is a provision of Flexible, Open and Distance Education as an alternative pathway of formal education.

The Course promotes Papua New Guinea values and beliefs which are found in our constitution, Government policies and reports. It is developed in line with the National Education Plan (2005 – 2014) and addresses an increase in the number of school leavers which has been coupled with a limited access to secondary and higher educational institutions.

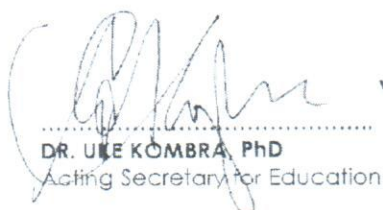
Flexible, Open and Distance Education is guided by the Department of Education's Mission which is fivefold:

- to facilitate and promote integral development of every individual
- to develop and encourage an education system which satisfies the requirements of Papua New Guinea and its people
- to establish, preserve, and improve standards of education throughout Papua New Guinea
- to make the benefits of such education available as widely as possible to all of the people
- to make education accessible to the physically, mentally and socially handicapped as well as to those who are educationally disadvantaged.

The College is enhanced to provide alternative and comparable path ways for students and adults to complete their education, through one system, many path ways and same learning outcomes.

It is our vision that Papua New Guineans harness all appropriate and affordable technologies to pursue this program.

I commend all those teachers, curriculum writers and instructional designers, who have contributed so much in developing this course.



DR. UKE KOMBRA, PhD
Acting Secretary for Education



UNIT 2: TRIGONOMETRY AND VECTORS

Introduction

Trigonometry is one of the oldest and most useful tools in mathematics. Trigonometry is used in surveying, forestry, engineering, modelling periodic phenomena, and the list goes on. We can look at trigonometry in three distinct but highly interconnected ways in this unit. We can consider trigonometry as a relation of the sides of a right triangle, called right triangle trigonometry. Students will come to understand angle measurement in both degrees and radians and will be provided with experiences in applying the six trigonometric functions (including inverse functions) as functions of angles in standard position.

Furthermore, in this unit, students will expand their understanding of trigonometry with the identities. Students will in detail work with fundamental identities, proving identities, double angle formula, half angle formulas to use the knowledge to solve related problems. Finally they will come to understand vectors with the basic concepts in trigonometry and will be provided with experiences in applying the problems involving vectors. The trigonometric concepts will be used to explore more practical solutions for vectors.



LEARNING OUTCOMES

On successful completion of this module, students will be able to:

- convert radian measures to degree and vice versa
- discuss quadrants and apply related angles
- apply sine and cosine rules
- sketch the basic and further trigonometric functions
- find solution of trigonometric equations
- apply Pythagoras' Theorem as the basis to form the basic trigonometric identities
- identify physical quantities as scalar and vector quantities
- represent equal and opposite vectors diagrammatically
- identify unit and position vectors
- solve vector addition problems
- apply scalar products of two vectors
- calculate angles between two vectors
- apply vector geometry in real life problems
- solve application problems involving triangle velocities.



TIME FRAME

This unit should be completed within 10 weeks.

If you set an average of 3 hours per day, you should be able to complete the unit comfortably by the end of the assigned week.

Try to do all the learning activities and compare your answers with the ones provided at the end of the unit. If you do not get a particular exercise right in the first attempt, you should not get discouraged but instead, go back and attempt it again. If you still do not get it right after several attempts then you should seek help from your friend or even your tutor. Do not pass any question without solving it first.

12.2.1 TRIGONOMETRY

Trigonometry is a branch of mathematics concerned with the lengths of the sides and size of the angles of triangles. Because many physical questions can be framed in terms of triangles, trigonometry has found wide use in the physical sciences and engineering.

Trigonometry is needed in surveying and architecture since relationships between lengths and angles in triangles are used directly.

Trigonometry is also used in electrical engineering. The functions that relate angles and side lengths in right triangles turn out to also be useful in expressing how AC electric current varies with time.

In geometry

In its geometry application, it is mainly used to solve triangles, usually right triangles. That is, given some angles and side lengths, we can find some or all the others.

Analytically

In a more advanced use, the trigonometric ratios such as Sine and Tangent are used as functions in equations and manipulated using algebra. In this way, it has many engineering applications such as electronic circuits and mechanical engineering. In this analytical application, it deals with angles drawn on a coordinate plane, and can be used to analyse things like motion and waves.

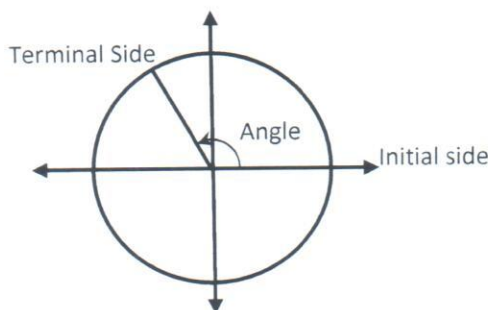
In this unit we will be mainly focusing on more advance use of trigonometric ratios and the use of trigonometric functions, trigonometric identities and vectors. The problems relating to the units involve analytical and critical thinking.



12.2.1.1: Radian Measures of Angles

Trigonometry means “measurement of triangles.” The name comes from Greek **trigonon** "triangle" + **metron** "measure").

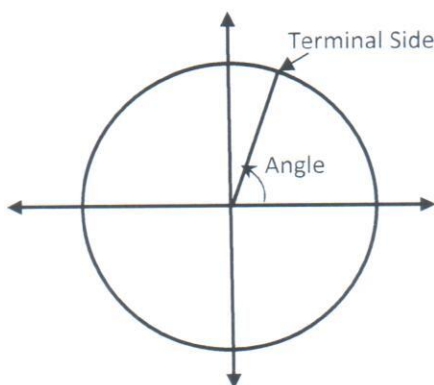
An **angle** is determined by rotating a ray (half-line) about its endpoint.



An **angle** is in standard position if its vertex is located at the origin and one ray is on the positive x-axis. The ray on the x-axis is called the **initial side**. It is the ray/line where the measurement of an n starts. The other ray is called the **terminal side**. It is where the measurement of an **angle** stops.

There are different types of angle formed by rotating the terminal side.

1. **Standard angle** is the angle between the positive x –axis and the terminal side/arm.
2. **Principal angle** is the angle between 0° and 360° .
3. **Co- terminal angle** is an angle less than 0° or greater than 360° that shares the same position as a principal angle.
4. **Negative angles** are angles obtained from rotating the terminal side/arm clockwise.
5. **Reference angle** is an angle between the terminal side/arm and x – axis.



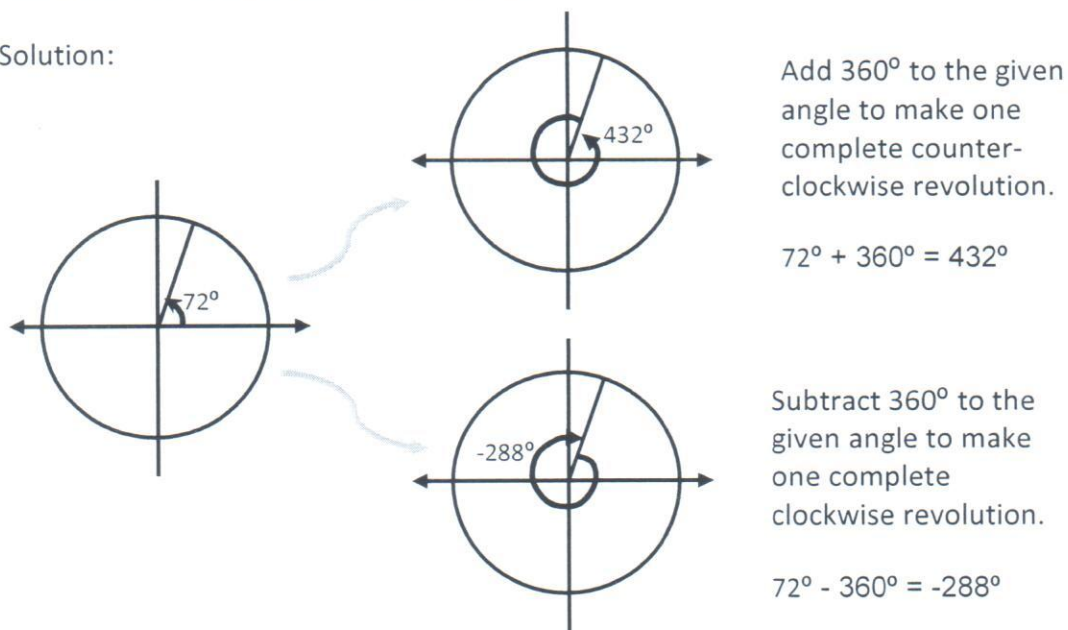
Finding the positive and negative co-terminal angles of a given angle

- To find the positive co-terminal angle, add 360° to the given angle to make one complete counter-clockwise revolution.
- To find the negative co-terminal angle subtract 360° to make one complete clockwise revolution.

**Example 1**

Find both the positive and negative co-terminal angles for 72° .

Solution:



The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side.

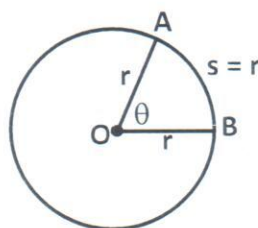
One way to measure angles is in **radians**.

To define a radian, you can use a **central angle** of a circle, one whose vertex is the centre of the circle.

Definition of radian

A radian is the measure of an angle θ that, when drawn as a central angle, subtends an arc whose length equals the length of the radius r of the circle.

Algebraically this means that $\theta = \frac{r}{s}$, where θ is measured in radians.





The following radian measures with their degree equivalents come up frequently so you should know.

$$180^\circ = \frac{1}{2} \text{ revolution} = \frac{2\pi}{2} = \pi \text{ radian}$$

$$90^\circ = \frac{1}{4} \text{ revolution} = \frac{2\pi}{4} = \frac{\pi}{2} \text{ radian}$$

$$60^\circ = \frac{1}{6} \text{ revolution} = \frac{2\pi}{6} = \frac{\pi}{3} \text{ radian}$$

$$45^\circ = \frac{1}{8} \text{ revolution} = \frac{2\pi}{8} = \frac{\pi}{4} \text{ radian}$$

$$30^\circ = \frac{1}{12} \text{ revolution} = \frac{2\pi}{12} = \frac{\pi}{6} \text{ radian}$$

Degree Measure

A second way to measure angles is in terms of **degrees**, denoted by the symbol $^\circ$.

A measure of one degree (1°) is equivalent to a rotation of $\frac{1}{360^\circ}$, a complete revolution about the vertex.

$$1^\circ = \frac{\pi}{180} \text{ rad}$$

$$1 \text{ rad} = \left(\frac{180^\circ}{\pi} \right)$$

When no units of angle measure are specified, **radian measure is implied**.

Conversion between Degrees and Radian Measures

1. Degrees to radians

To convert degrees to radians measure, multiply $\frac{\pi}{180^\circ}$.

Example 1

Change 48° to radians.

Solution:

$$180^\circ = \pi \text{ radians}$$

$$\text{Therefore, } 48^\circ = \frac{48 \times \pi}{180} = 0.838 \text{ radians}$$

**Example 2**

Convert 120° to radians.

Solution: 120° is a third of 360° , which is 2π .

$$\text{Therefore, } 120^\circ = \frac{120 \times \pi}{180} = \frac{2\pi}{3} = 2.094 \text{ radians}$$

2. Radians to Degrees

To convert radians to degree measure, multiply $\frac{180^\circ}{\pi}$.

Example 1

Change $\frac{2\pi}{15}$ radians to degrees.

Solution:

$$\frac{2\pi}{15} \text{ radians} = \frac{2\pi}{15} \times \frac{180^\circ}{\pi} = \frac{2 \times 180^\circ}{15} = \frac{360^\circ}{15} = 24^\circ$$

Example 2

Convert $\frac{9\pi}{2}$ radians to degrees.

Solution:

$$\frac{9\pi}{2} \text{ radians} = \frac{9\pi}{2} \times \frac{180^\circ}{\pi} = \frac{9 \times 180^\circ}{2} = \frac{1620^\circ}{2} = 810^\circ$$

Now do the learning activity on the next page.

**LEARNING ACTIVITY 12.2.1.1**

20 minutes

1. Convert the following degree measures to radians:

a. 270°

b. 210°

c. 135°

d. 300°

e. 36°

2. Convert the following radian measures to degrees:

a. $\frac{\pi}{2}$

b. $\frac{7\pi}{4}$

c. 2 radians

d. $\frac{5\pi}{6}$

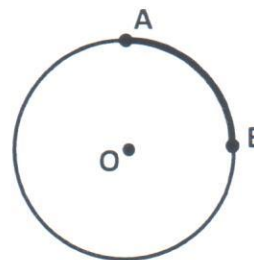
e. $\frac{7\pi}{9}$



12.2.1.2: The Length of an Arc

An **arc** of a circle is a "portion" of the circumference of the circle.

The **length of an arc** is simply the length of its "portion" of the circumference. Actually, the circumference itself can be considered an arc length.



An angle whose vertex is the centre of a circle and whose sides pass through a pair of points on the circle is called a **central angle**.

Each central angle forms an **arc length** between the points that it passes through on the circle. The arc length of a circle can be found by first calculating what fraction of the circle is represented by the central angle.

Then you find the circumference of the circle. Finally, you find the fraction of that circumference.

Example

A circle has a radius of 10 cm and a central angle of 60° . Calculate the arc length for that angle.

Solution: We must first determine what fraction of the circle is represented by the central angle 60° . Since there are 360° in a full circle, we can find the fraction of a circle by simplifying $\frac{60}{360}$. $\frac{60}{360} = \frac{1}{6}$

The sector represents $\frac{1}{6}$ of the circle.

To find the arc length, we now need to find the circumference of the circle. Circumference of a circle is equal to $2\pi r$.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi \times 10 \text{ cm} \\ &= 62.8 \text{ cm} \end{aligned}$$

The circumference of this circle is about 62.8 cm. To find the arc length, we need to find $\frac{1}{6}$ of 62.8 cm.

$$\text{Hence, } \frac{1}{6} \times 62.8 = 10.5 \text{ cm}$$

The arc length formed by this central angle is about **10.5 cm**.



To make the process quicker, you can refer to the formula used to determine the arc length formed by a central angle:

$$C_A = \frac{\theta}{360^\circ} \times 2\pi r, \text{ where } C_A \text{ is the arc length, and } r \text{ is the radius of the circle.}$$

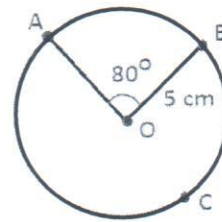
Now let us have the following examples.

Example 1

Find the lengths of major and minor arcs **AB** correct to the tenth of a cm. The minor arc **AB** has a central angle which measures 80° .

Solution: radius of the circle (r) = 5 cm

Using the formula: $C_A = \frac{\theta}{360^\circ} \times 2\pi r$,



- a. Find the length of minor arc $\widehat{AB}$.

$$\begin{aligned} C_A &= \frac{\theta}{360^\circ} \times 2\pi r \\ &= \frac{80^\circ}{360^\circ} \times 2 \times \pi \times 5 \text{ cm} \\ &= \frac{800\pi}{360} \text{ cm} \\ &= \frac{20\pi}{9} \text{ cm} \\ &= 6.977 \text{ or } \mathbf{7.0 \text{ cm}} \end{aligned}$$

- b. To find the length of the major arc $\widehat{AB}$ we need to use the reflex angle.

Thus we have, $\theta = 360^\circ - 80^\circ = 280^\circ$

$$\begin{aligned} C_A &= \frac{\theta}{360^\circ} \times 2\pi r \\ &= \frac{280^\circ}{360^\circ} \times 2 \times \pi \times 5 \text{ cm} \\ &= \frac{2800\pi}{360} \text{ cm} \\ &= \frac{70\pi}{9} \text{ cm} \\ &= \mathbf{24.4 \text{ cm}} \end{aligned}$$

**Example 2**

Find the length of a quadrant of a circle whose diameter is 14 centimetres.

Solution:

The central angle of a quadrant = $\theta = 90^\circ$

Use the formula, $C_A = \frac{\theta}{360^\circ} \times 2\pi r$

The diameter of the circle is given. Hence, $2\pi r$ can be substitute with πD .

Thus we have, $C_A = \frac{\theta}{360^\circ} \times \pi D$.

$$\begin{aligned}\text{Substitute } D = 14 \text{ cm, } \theta = 90^\circ \text{ we get } C_A &= \frac{90^\circ}{360^\circ} \times 14\pi \text{ cm} \\ &= \frac{1}{4} \times 14\pi \text{ cm} \\ &= \mathbf{10.99 \text{ cm or } 11 \text{ cm}}\end{aligned}$$

**LEARNING ACTIVITY 12.2.1.2**

20 minutes

1. In a circle, a central angle of $\frac{1}{3}$ radians subtends an arc of 3 centimetres.
Find the length, in centimetres, of the radius of the circle.
2. The number of radians in a complete rotation is approximately ____.
3. Find, in centimetres, the length of an arc intercepted by a central angle of 4 radians in a circle with a radius of 3.5 cm.
4. A sector has a radius of 12 cm. and an angle of 65° . To the nearest tenth of a cm, find its arc length.

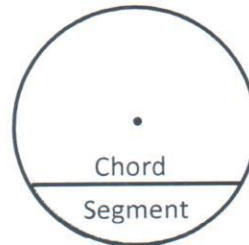
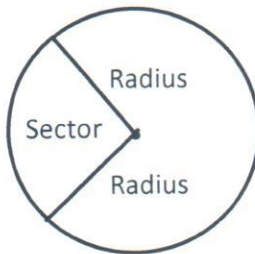


12.2.1.3: The Area of a Sector and Segment of a Circle

In this section, we will learn the area of a sector in a circle, the formula for area of sector (in degrees), the formula for area of sector (in radians), how to calculate the central angle of a sector, how to calculate the radius of a sector and how to calculate the area of a segment.

There are two main "slices" of a circle:

- The "pizza" slice is called a **Sector**.
- And the slice made by a chord is called a **Segment**.



The Sector of a Circle

The **sector of a circle** is a pie-shaped region bounded by two radii and an arc lying between the radii.

Area of Sectors of Circles

For a **Semicircle** (half a circle)



$$A = \frac{1}{2} \pi r^2$$

For a **Quadrant** (one-fourth of a circle)



$$A = \frac{1}{4} \pi r^2$$

For **any Sector of a circle** (fractional part of the area of a circle)

1. $A = \frac{n}{360^\circ} \pi r^2$, where n is the number of degrees in the central angle of the sector.
2. $A = \frac{C_A}{2\pi r} \pi r^2$, where C_A is the arc length of the sector.

Notice that when finding the area of a sector, you are actually finding a fractional part of the area of the entire circle. The fraction is determined by the ratio of the central angle of the sector to the "entire central angle" of 360 degrees, or by the ratio of the arc length to the entire circumference. The second formula can be algebraically reduced, but it is easier to remember that you are dealing with fractional parts.



The area of a sector is a fractional part of the area of the circle. This area is proportional to the central angle. In other words, the bigger the central angle, the larger is the area of the sector.

Finding the areas of sectors of a circle

If you know the radius of a circle and a central angle, you can determine the sector area for that angle by first calculating the fraction of the circle that the sector represents, then finding the area of the entire circle, and finally calculating that fraction of the total area.

For example, if we again use a circle with a radius of 10 cm and a central angle of 60° , we can determine the sector area by first finding the fraction of the circle, which we calculated earlier and then calculating the area of the entire circle.

The area of a circle is equal to πr^2 . The area of this circle can be found using this formula.

$$A = \pi r^2$$

$$A = 3.14(10 \text{ cm})^2$$

$$A = 314 \text{ cm}^2$$

Now we need to find $\frac{1}{6}$ of 314 cm^2 to determine the area of the sector formed by this central angle.

$$\frac{1}{6} \times 314 = 52.3 \text{ cm}^2$$

The area of the sector formed by central angle is about **52.3 cm^2** .

Now look at the following examples.

Example 1

Calculate the arc length and the area of a sector formed by a 30° central angle on a circle with a radius of 2 cm.

Solution:

First, calculate the arc length. To do this first determine what fraction of the circle is represented by the central angle 30° . Since there are 360° in a full circle, find the fraction of a circle by simplifying $\frac{30^\circ}{360^\circ}$.

$$\frac{30}{360}, \frac{30}{360} = \frac{1}{12}$$

The sector represents $\frac{1}{12}$ of the circle.



To find the arc length, find the circumference of the entire circle. Circumference of a circle is equal to $2\pi r$.

$$\begin{aligned} C &= 2\pi r \\ &= 2\pi \times 2 \\ &= 2 \times 3.14 \times 2 \\ &= 12.56 \text{ cm} \end{aligned}$$

The circumference of this circle is about 12.56 cm.

Now, find the arc length, find $\frac{1}{12}$ of 12.56 cm. $\frac{1}{12} \times 12.56 = 1.047 \text{ cm}$

The arc length formed by this central angle is about 1.047 cm.

Next, determine the area of the sector by calculating the area of the entire circle and then multiplying by $\frac{1}{12}$. The area of a circle is equal to πr^2 . The area of this circle then can be found using this formula.

$$\begin{aligned} A &= \pi r^2 \\ A &= 3.14(2 \text{ cm})^2 \\ &= 3.14 \times 4 \text{ cm}^2 \\ &= 12.56 \text{ cm}^2 \end{aligned}$$

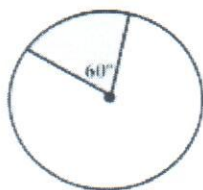
Now, find $\frac{1}{12}$ of 12.56 cm² to determine the area of the sector formed by this central angle.

$$\frac{1}{12} \times 12.56 = 1.047 \text{ cm}^2$$

The area of the sector formed by this central angle is about 1 cm².

Example 2

Find the area of a sector with a central angle of 60 degrees and a radius of 10. Express answer to the nearest tenth. Use $\pi = 3.1416$



Solution:

$$\begin{aligned} A &= \frac{n}{360^\circ} \pi r^2 \\ A &= \frac{60^\circ}{360^\circ} (3.14)(10)^2 \\ &= \frac{1}{6} (3.14)(100) \\ &= \frac{1}{6} (314) \\ &= 52.36 \\ &= \mathbf{52.4} \end{aligned}$$

**Example 3**

Find the area of a sector with an arc length of 40 cm and a radius of 12 cm. Use $\pi = 3.14$

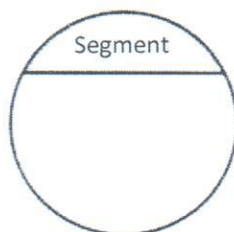
Solution:

$$A = \frac{C_A}{2\pi r} \pi r^2$$
$$A = \frac{40 \text{ cm}}{2(12 \text{ cm})\pi} \pi (12 \text{ cm})^2$$
$$= \frac{40}{24\pi} (144\pi)$$
$$= (40)(6 \text{ cm}^2)$$
$$= 240 \text{ cm}^2$$

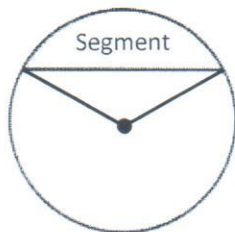
Segment of a circle

The **segment of a circle** is the region bounded by a chord and the arc subtended by the chord.

While a sector looks like a "pie" slice, a **segment** looks like the "pie" slice with the triangular portion cut off. The segment is only the small partially curved figure left when the triangle is removed.



Dealing with the area of a segment is very similar to working with the area of a sector. If you find the area of the sector and subtract the area of the triangle, you will have the area of the segment portion of the circle.



Formula for the area of a segment:

Area of a Segment = Area of a Sector – Area of the Triangle

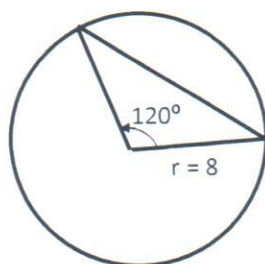
$$A_{\text{Segment}} = A_{\text{Sector}} - A_{\text{triangle}}$$



Let's look at an example problem.

Example 1

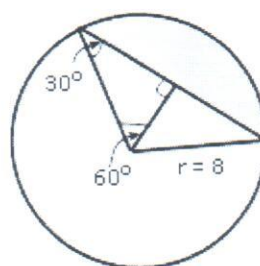
Find the area of a segment of a circle with a central angle of 120 degrees and a radius of 8. Express answer to two decimal places..



Solution:

- a) Find the area of the Sector.

$$A = \frac{120}{360} \pi 8^2 = \frac{1}{3} \pi (64) = \frac{64}{3} \pi = \mathbf{67.02}$$



- b) Find the area of the segment.

To find the area of the segment, we need to find the area of the triangle.

Drawing the altitude forms a 30-60-90 degree triangle as shown above.

Using trigonometry (or the 30-60-90 rules), find the altitude.

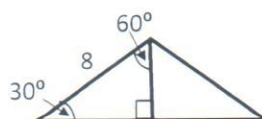
$$\cos \theta = \frac{a}{h}$$

$$\cos 60^\circ = \frac{a}{8}$$

$$a = 8 \cos 60^\circ$$

$$a = 8 (0.5)$$

$$a = 4$$



Now, using the Pythagorean Theorem, find the other leg.

$$b = \sqrt{8^2 - 4^2}$$

$$b = \sqrt{64 - 16}$$

$$b = \sqrt{48}$$

$$b = \sqrt{(16)(3)}$$

$$b = 4\sqrt{3} = 6.92820323 = 6.928$$

Hence the area of the triangle is: $A = \frac{1}{2}bh$

$$= \frac{1}{2} (2)(6.928)(4) = \mathbf{27.71}$$



Now we can find the area of the segment.

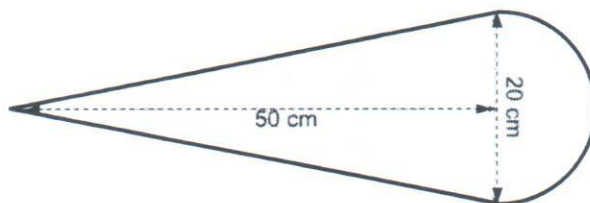
$$A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}$$

$$A_{\text{segment}} = 67.02 - 27.71$$

$$= 39.31 \text{ square units}$$

Example 2

A metal tool consists of a semicircle and an isosceles triangle joined together as shown.



What area of metal is needed to make the tool? Use 3.14 as an approximation for π .

Solution:

- a. Find the area of the semicircle first:

Use the formula for the area of a circle: $A = \pi r^2$ where r is the radius

Therefore, the formula for the area of a semicircle is $A = \frac{1}{2} \pi r^2$

Given: $r = 10$ and $\pi = 3.14$

$$\text{So } A = \frac{1}{2} (3.14)(10)^2 = \frac{1}{2} (3.14)(100) = 157$$

- b. Next find the area of the triangle using the formula $A = \frac{1}{2} bh$

$b = 20$ and $h = 50$

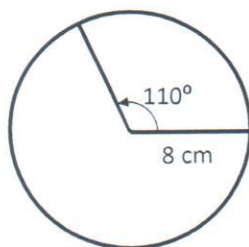
$$\text{So } A = \frac{1}{2} (20)(50) = 500$$

$$\text{So the total area of metal used} = 157 \text{ cm}^2 + 500 \text{ cm}^2 = 657 \text{ cm}^2$$

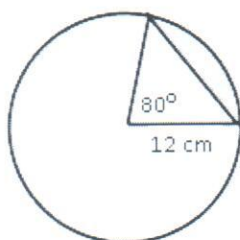
**LEARNING ACTIVITY 12.2.1.3**

20 minutes

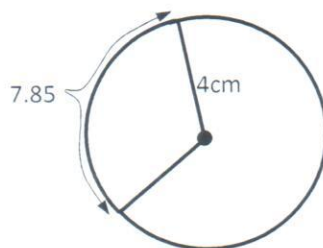
1. Find the area of the sector of a circle with radius 8 cm formed by a central angle of 110° .



2. Find the area of the shaded region in the circle with radius of 12 cm and a central angle of 80° .

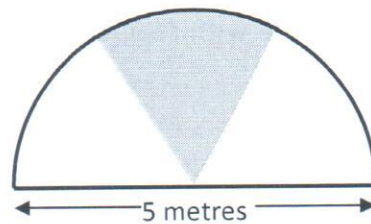


3. Find the area of the shaded sector shown at the left. The radius of the circle is 4 units and the length of the arc (the curved edge of the sector) measures 7.85 units. Express answer to the nearest tenth of a square unit.

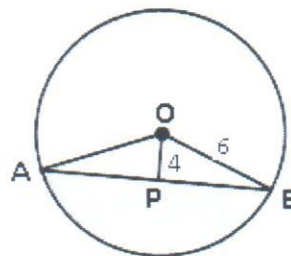




4. A cathedral window is built in the shape of a semicircle. If the window is to contain three stained glass sections of equal size, what is the area of each stained glass section? Express answer to the nearest square metre.



5. Find the area of the shaded segment of circle O to the nearest tenth of a square unit. The radius of the circle is 6 units. The perpendicular from O to the chord joining point A to point B measures 4 units.



6. Given a sector with radius $r = 3$ and a corresponding arc length of 5π , find the area of the sector.



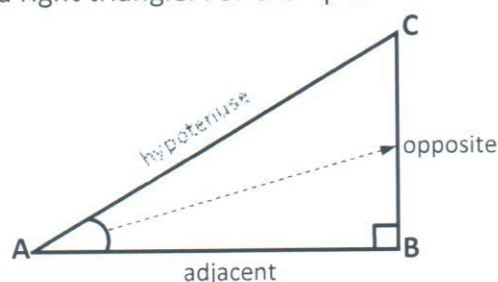
12.2.1.4: The Trigonometric Functions and Their Graphs

This section includes the study of the basic trigonometric functions and instructions for graphing processes of basic functions, amplitude shifts, horizontal shifts, and vertical shifts of trigonometric functions.

The functions (also called the **circular functions**) comprising trigonometry: the **cosecant** ($\csc x$), **cosine** ($\cos x$), **cotangent** ($\cot x$), **secant** ($\sec x$), **sine** ($\sin x$), and **tangent** ($\tan x$).

Let us start by recalling what you have learned about trigonometric ratios in Grade 11 Mathematics.

The six basic trigonometric ratios (sine, cosine, tangent, secant, cosecant and cotangent) are defined using the sides of a right triangle. For example:



$$\text{Sine } A = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\text{Cosecant } A = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\text{Cosine } A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{Secant } A = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\text{Tangent } A = \frac{\text{opposite}}{\text{adjacent}}$$

$$\text{Cotangent} = \frac{\text{adjacent}}{\text{opposite}}$$

Notice that the three new ratios at the right are reciprocals of the ratios on the left. Applying a little algebra shows the connection between these ratios.

$$\csc A = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{1}{\frac{\text{opposite}}{\text{hypotenuse}}} = \frac{1}{\sin A}$$

Reciprocal ratios	
$\sin A = \frac{1}{\csc A}$	$\csc A = \frac{1}{\sin A}$
$\cos A = \frac{1}{\sec A}$	$\sec A = \frac{1}{\cos A}$
$\tan A = \frac{1}{\cot A}$	$\cot A = \frac{1}{\tan A}$

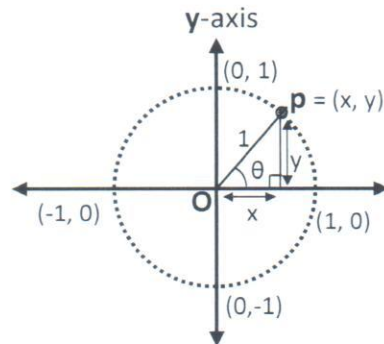
These definitions only apply to acute angles (angles between 0° and 90°). This is because the other two angles in a right triangle are acute.



In this section, we will learn the definitions for the basic functions which apply to any angle. A Greek letter (theta θ) will now be used to represent the angle.

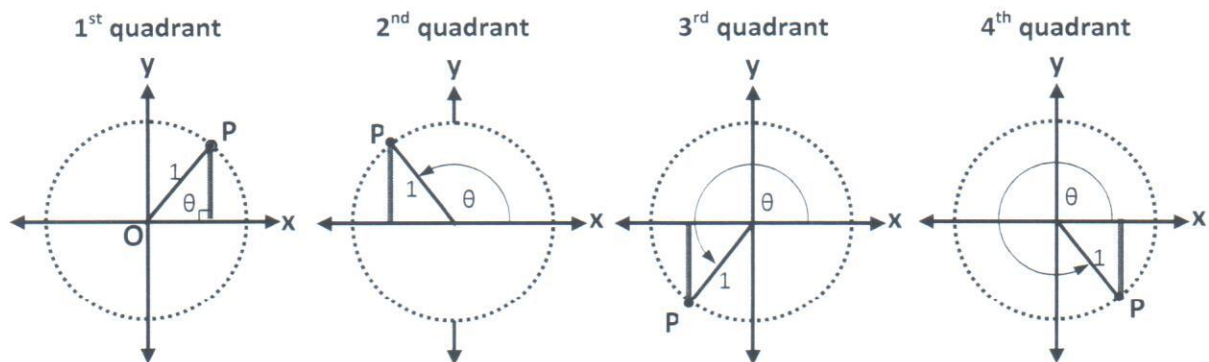
12.2.1.4.1 The Sine, Cosine and Tangent Functions of Any Angle

Sine, cosine and tangent can be defined using coordinates. These definitions apply to any angle. In the figure below, **OP** is a line one (1) unit long and **P** has coordinates (x, y) .

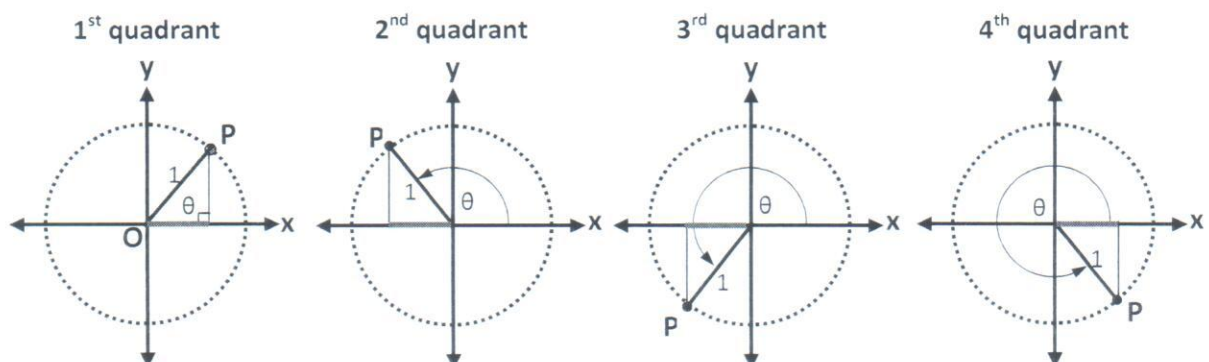


As **OP** rotates about the origin **O** in a positive direction (anticlockwise) it makes an angle θ with the positive x -axis. Whatever the size of θ , the coordinates of **P** give the trigonometric ratios of θ .

Shown in red in each quadrant in the diagram below is the $\sin \theta = y$ -coordinate of **P**.

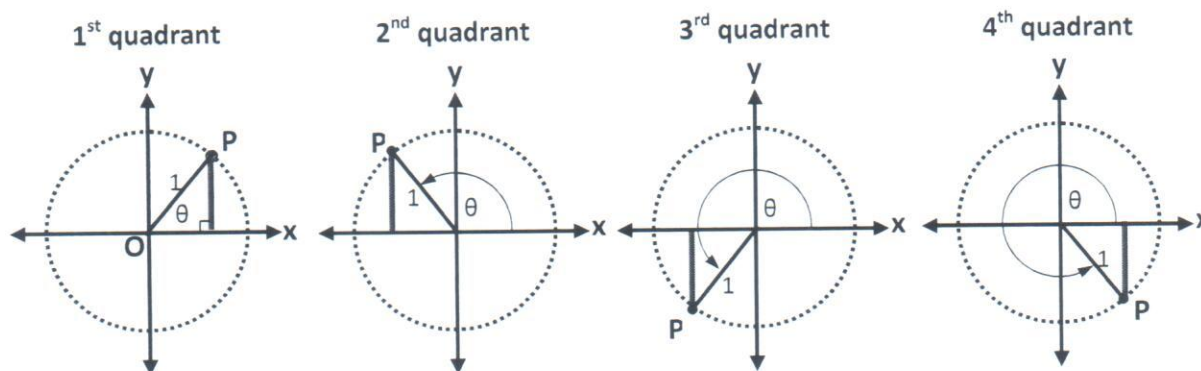


Shown in red in each quadrant in the diagram below is the $\cos \theta = x$ -coordinate of **P**.





$\tan \theta = \frac{y - \text{coordinate of } P}{x - \text{coordinate of } P}$ is shown in red in each quadrant.

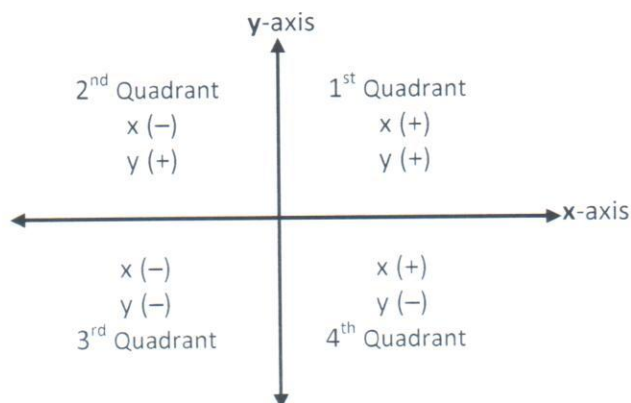


In summary, if we let θ be any real number and let $P(x, y)$ be the terminal point on the unit circle determined by θ . We defined the trigonometric functions as shown in the table below.

The Trigonometric Functions	
$\sin \theta = y$	$\csc \theta = \frac{1}{y} \quad (y \neq 0)$
$\cos \theta = x$	$\sec \theta = \frac{1}{x} \quad (x \neq 0)$
$\tan \theta = \frac{y}{x} \quad (x \neq 0)$	$\cot \theta = \frac{x}{y} \quad (y \neq 0)$

As the point P rotates, its coordinates are **positive** or **negative** depending upon which quadrant P is in. So the trigonometric ratios of angle θ will be positive or negative depending upon which quadrant it is in.

See figure on the next page.



The table on the next page shows the proper signs of the Trigonometric (Circular) functions in the four (4) quadrants for which θ lies.



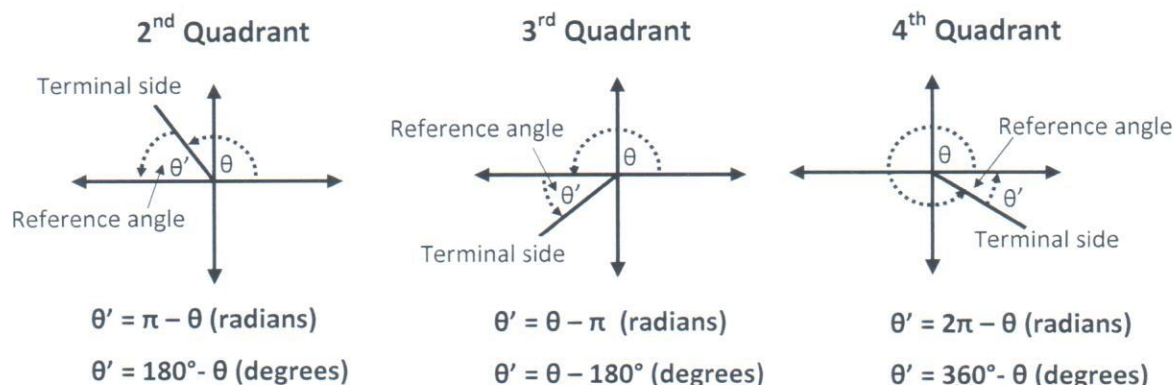
THE SIGNS OF TRIGONOMETRIC FUNCTIONS

Quadrants Functions	I	II	III	IV
$\sin \theta$	+	+	-	-
$\cos \theta$	+	-	-	+
$\tan \theta$	+	-	+	-
$\cot \theta$	+	-	+	-
$\sec \theta$	+	-	-	+
$\csc \theta$	+	+	-	-

The Reference angle

Reference angle is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

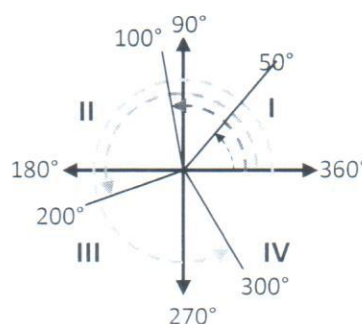
The reference angles for θ in Quadrants II, III, and IV.



Example 1

What is the sign (+ or -) of the following?

- $\sin 50^\circ$
- $\cos 100^\circ$
- $\tan 200^\circ$
- $\csc 300^\circ$



Solution:

- Positive** (50° is in the first quadrant - all are positive)
- Negative** (100° is in the second quadrant)
- Positive** (200° is in the 3rd quadrant, and tan is positive there).
- Negative** (300° is in the 4th quadrant and sin is negative there, so it follows that csc will also be negative).



Example 2

Find the reference angle θ' of the following:

- a. $\theta = 150^\circ$ b. $\theta = 225^\circ$ c. $\theta = 300^\circ$ d. $\theta = 2.3$

Solution:

- a. 150° lies in 2nd Quadrant. So, $\theta' = 180^\circ - \theta = 180^\circ - 150^\circ = 30^\circ$
b. 225° is in the 3rd Quadrant. So, $\theta' = \theta - 180^\circ = 225^\circ - 180^\circ = 45^\circ$
c. 300° is in the 4th Quadrant. So, $\theta' = 360^\circ - \theta = 360^\circ - 300^\circ = 60^\circ$
d. $2.3 = 131.78^\circ$ which lies between 90° and 180° . It follows that it is in 2nd Quadrant and its reference is $\theta' = 180^\circ - \theta = 180^\circ - 131.78^\circ = 48.22^\circ$

**LEARNING ACTIVITY 12.2.1.4.1**

20 minutes

1. In which quadrant is θ if $\cos \theta$ is negative and $\sin \theta$ is positive?

2. What is the sign (+ or -) of the following:

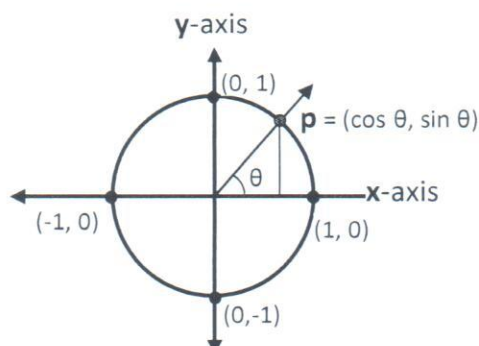
- a. $\sin 100^\circ$
b. $\cos 188^\circ$
c. $\sec (-15^\circ)$
d. $\tan 225^\circ$

3. Find the reference angle θ' of the following:

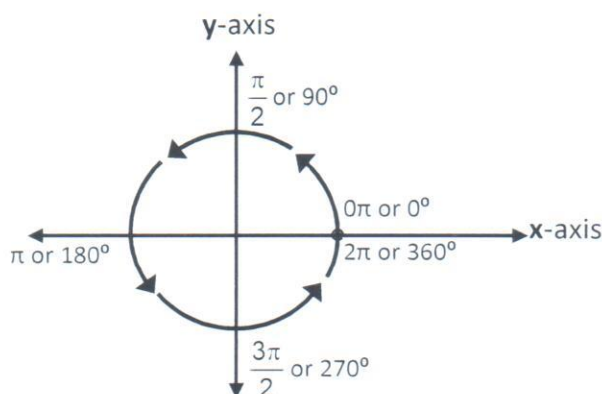
- a. $\theta = 120^\circ$
b. $\theta = 210^\circ$
c. $\theta = 330^\circ$
d. $\theta = 2.5$

12.2.1.4.2 The Unit Circle and the Values of Trigonometric Functions

The **unit circle** is a circle with a radius that equals 1. The angle θ is formed from the ray extending from the origin through a point on the unit circle and the **x-axis**; see diagram below. The value of **sin θ** equals the **y-coordinate** of the point **p** and the value of **cos θ** equals the **x-coordinate** of the point **p** as shown in the diagram below.



This unit circle below shows the measurements of angles in radians and degrees. Beginning at 0π , follow the circle counter-clockwise. As angle θ increases to $\frac{\pi}{2}$ radians or 90° , the value of cosine (the x-coordinate) decreases because the point is approaching the y-axis while the value of Sine (the y-coordinate) increases. When one counter-clockwise revolution has been completed, the point has moved 360° or 2π .



Evaluating Trigonometric Functions of any angle

To evaluate trigonometric function of any angle θ :

1. Determine the function value of the associated reference angle θ' .
2. Depending on the quadrant in which θ lies, affix the appropriate sign to the function value.

**Example 1**

Find the value of the function $\sin 330^\circ$ using the reference angle.

Solution: Since $\theta = 330^\circ$ lies in the 4th Quadrant, the reference angle is:

$$\theta' = 360^\circ - \theta = 360^\circ - 330^\circ = 30^\circ$$

Moreover, the sine is negative in the 4th Quadrant (see the table on page 25), so

$$\sin 330^\circ = -\sin 30^\circ = -\frac{1}{2} \text{ or } -0.5$$

Example 2

Find the value of the function $\cos 120^\circ$ using the reference angle.

Solution: Since $\theta = 120^\circ$ lies in the 2nd Quadrant, the reference angle is:

$$\theta' = 180^\circ - \theta = 180^\circ - 120^\circ = 60^\circ$$

Moreover, the cosine is negative in the 2nd Quadrant (see the table on page 25), so

$$\cos 120^\circ = -\cos 60^\circ = -\frac{1}{2} \text{ or } -0.5$$

Example 3

Find the value of the function $\tan (-210^\circ)$ using the reference angle.

Solution: Because $-210^\circ + 360^\circ = 150^\circ$, it follows that -210° is co-terminal with the 2nd Quadrant angle 150° .

So, the reference angle is: $\theta' = 180^\circ - 150^\circ = 30^\circ$

Finally, because the tangent is negative in 2nd Quadrant, so

$$\tan (-210^\circ) = (-) \tan 30^\circ = -\frac{\sqrt{3}}{3}$$

Example 4

Using the reference angle, evaluate the function $\cos \frac{4\pi}{3}$.

Solution: Since $\theta = \frac{4\pi}{3}$ lies on Quadrant III, the reference angle is:

$$\theta' = \theta - \pi = \frac{4\pi}{3} - \pi = \frac{4\pi - 3\pi}{3} = \frac{\pi}{3}$$

Consulting the table on page 25, the cosine is negative in Quadrant III.

$$\text{So, } \cos \frac{4\pi}{3} = -\frac{\pi}{3} = -\frac{1}{2}$$

The table below shows the Values of the Common Trigonometric Angle of Measurement.



Common Trigonometric Angle Measurements

Angle in Degree	Angle in Radians	Sine	Cosine	Tangent	Cosecant	Secant	Cotangent
0°	0	0	1	0	ND	1	ND
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	1	0	ND	1	ND	0
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	-2	$-\frac{\sqrt{3}}{3}$
135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	2	$-\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
180°	π	0	-1	0	ND	-1	ND
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	-2	$-\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	-2	$\frac{\sqrt{3}}{3}$
270°	$\frac{3\pi}{2}$	-1	0	ND	-1	ND	0
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2\sqrt{3}}{3}$	2	$-\frac{\sqrt{3}}{3}$
315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$	-2	$\frac{2\sqrt{3}}{3}$	$-\sqrt{3}$
360°	2π	0	1	0	ND	1	ND

Note: ND means **not defined**.

Now look at the examples on the next page.



Example 5

Using the table on page 30, find the exact numerical value of the following expressions.

a) $\cos \frac{2\pi}{3} + \sin \frac{3\pi}{2}$

Solution: $\cos \frac{2\pi}{3} + \sin \frac{3\pi}{2} = -\frac{1}{2} + (-1)$
 $= -\frac{1}{2} - 1$

b) $\cos 60^\circ \cos 45^\circ + \sin 30^\circ \sin 45^\circ$

Solution: $\cos 60^\circ \cos 45^\circ + \sin 30^\circ \sin 45^\circ = \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$
 $= \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4}$
 $= \frac{2\sqrt{2}}{4}$
 $= \frac{\sqrt{2}}{2}$

**LEARNING ACTIVITY 12.2.1.4.2**

20 minutes

1. Find the values of the following functions using the reference angle.

a $\cos 135^\circ$

b $\tan 240^\circ$

c $\sin 150^\circ$



d $\csc (-30^\circ)$

2. Find the value of each function using the table on page 30.

a. $\tan \left(-\frac{\pi}{3}\right)$

b. $\cos \frac{2\pi}{3}$

c. $\sin \frac{19\pi}{4}$

3. Using the table on page 30, find the exact numerical value of the following expressions.

a. $\cos 60^\circ \sec 60^\circ + \sin 30^\circ \csc 30^\circ$

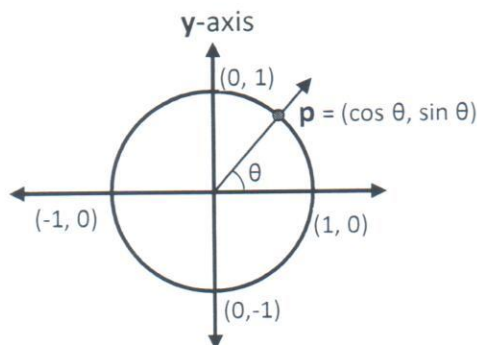
b. $\cot 30^\circ \csc 60^\circ - \tan 60^\circ \sec 150^\circ$

**12.2.1.4.3 The Graphs of Sine and Cosine Functions: $y = \sin x$ and $y = \cos x$**

There are two ways to prepare for graphing the basic sine and cosine functions in the form $y = \sin x$ and $y = \cos x$: evaluating the function and using the unit circle.

I. The Sine Function: $f(x)$ or $y = \sin x$

The zeros of $y = \sin x$ are at the multiples of π . And it is there that the graph crosses the x -axis, because there $\sin x = 0$. But what is the maximum value of the graph, and what is its minimum value?



$\sin x$ has a maximum value of 1 at $\frac{\pi}{2}$, and a minimum value of -1 at $\frac{3\pi}{2}$ and all angles co-terminal with them.

To evaluate the basic sine function, set up a table of values using the intervals 0π , $\frac{\pi}{2}$, π , $\frac{3\pi}{2}$, and 2π for x and calculating the corresponding y value.

 $f(x)$ or $y = \sin x$

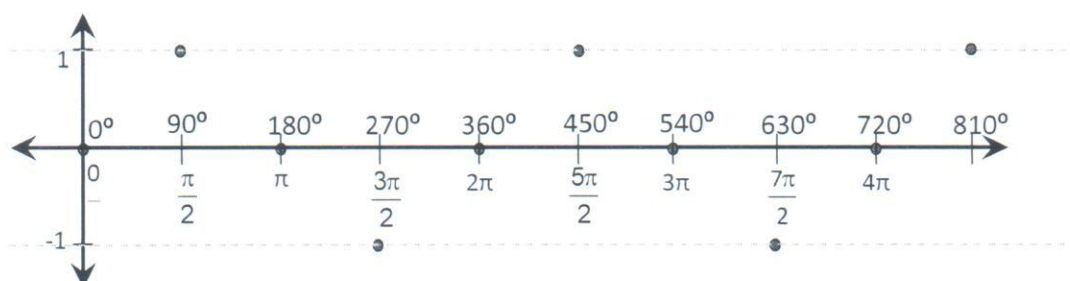
$f(x)$ or y	x
0	0π or 0°
1	$\frac{\pi}{2}$ or 90°
0	π or 180°
-1	$\frac{3\pi}{2}$ or 270°
0	2π or 360°

To use the unit circle, the x -coordinates remain the same as within the list above. To find the y -coordinate of the point to graph, first locate the point P on the unit circle that corresponds to the angle θ given by the x -coordinate. Then, use the y -coordinate of the point P as the y value of the point to graph.



Now use the information to draw the graph of the function sine or $y = \sin x$, label the x-axis with the values 0π , $\frac{\pi}{2}$, π , $\frac{3\pi}{2}$, and 2π . Then plot points for the value of $f(x)$ or y from either the table or the unit circle.

Other points may be added for the intermediate values between those listed above to obtain a more complete graph, and a best fit line can be drawn by connecting the points.

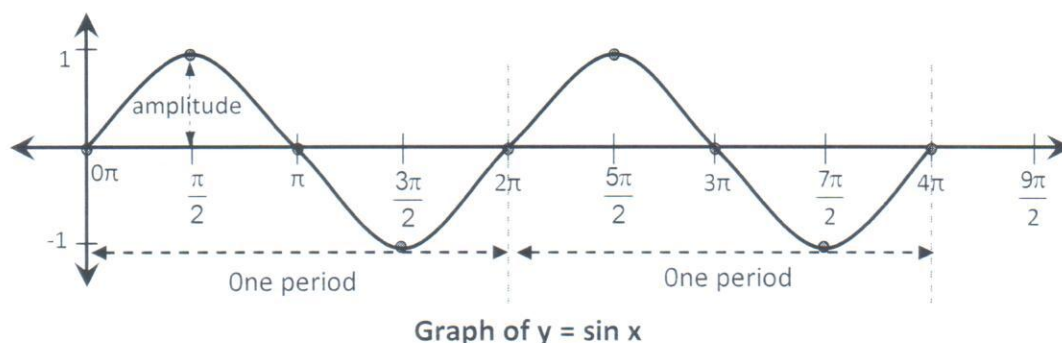


This function has **amplitude** of 1 because the graph goes one unit up and one unit down from the midline of the graph. This function has a **period** of 2π because the sine wave repeats every 2π units.

- The **amplitude** of a periodic function is one half the distance between the maximum value and its minimum value
- The **period** of a function is the horizontal length of one repeating pattern or interval.
- When the values of the function regularly repeat themselves, we say that the function is **periodic**.

Using the fact that the function is periodic with period 2π , we get the complete graph by continuing the same pattern to the left and to the right in every successive interval of length 2π .

The figure below is the completed graph showing two periods of the sine function.



The graph of the sine function is symmetrical with respect to the origin. This is expected, since sine is an odd function.



II. The Cosine Function: $f(x)$ or $y = \cos x$

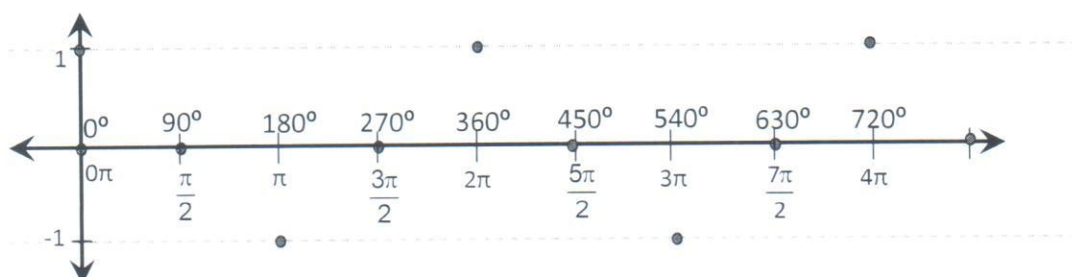
The graph of the cosine function $y = \cos x$ is drawn in a similar manner as the sine function. Using a table of values:

$f(x)$ or $y = \cos x$

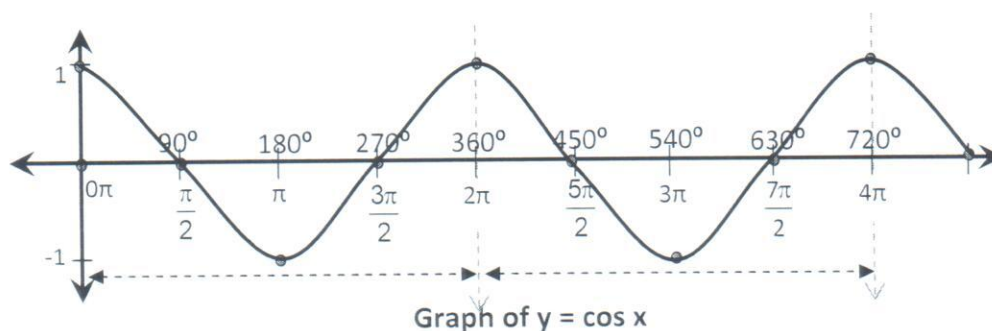
$f(x)$ or y	x
0	0π or 0°
1	$\frac{\pi}{2}$ or 90°
0	π or 180°
-1	$\frac{3\pi}{2}$ or 270°
0	2π or 360°

To use the unit circle, the x -coordinate remains the same as the list on the previous page. To find the y -coordinate of the point to graph, first locate the point p on the unit circle that corresponds to the angle θ given by the x -coordinate. Then, use the x -coordinate of the point p as the y value of the point to graph.

To draw the graph of one period of cosine or $y = \cos x$, label the x -axis with the values 0π , $\frac{\pi}{2}$, π , $\frac{3\pi}{2}$, and 2π . Then plot points for the value of $f(x)$ or y from either the table or the unit circle.



Add other points as required for the intermediate values between those above to obtain a more complete graph, and draw a best fit line connecting the points. The graph below shows two period of the cosine function.



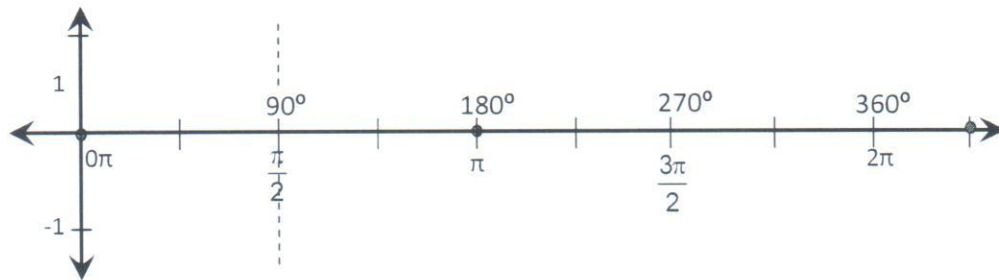


The graph of the Tangent Function: $y = \tan x$

The tangent value at an angle θ is equal to the sine value divided by the cosine value of the same angle θ .

$$\text{Tangent } \theta = \frac{\text{Sine value}}{\text{Cosine value}}$$

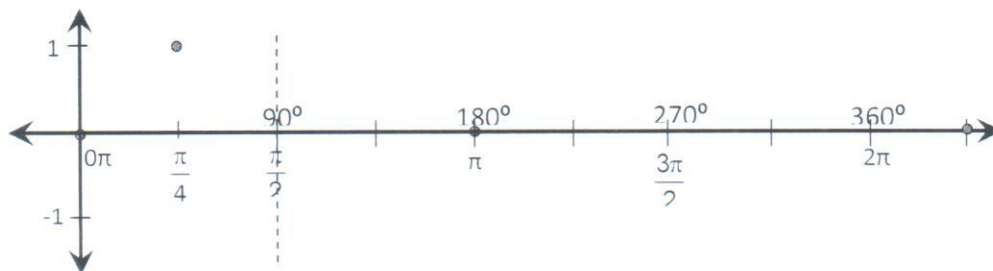
The value of tangent at 0π for the unit circle is $\frac{0}{1}$, which is equivalent to 0. The value of tangent at $\frac{\pi}{2}$ is $\frac{1}{0}$, which is undefined. Therefore the tangent function is undefined or does not exist at $\frac{\pi}{2}$. This is illustrated by drawing an **asymptote** (vertical dashed line) at $\frac{\pi}{2}$. See the figure below.



The value of tangent at π is $\frac{0}{1}$, which results in 0. To determine how the tangent behaves between 0π and the asymptote, find the sine and cosine values of $\frac{\pi}{4}$, which is half way between 0π and $\frac{\pi}{2}$. Consulting the **Common Trigonometric Angle measurements**, the tangent of $\frac{\pi}{4}$ is $\frac{\sqrt{2}}{2}$ (sine) divided by $\frac{\sqrt{2}}{2}$ (cosine). Solving the tangent of $\frac{\pi}{4}$, we have

$$\text{Tangent of } \frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{2}}{2} \times \frac{2}{\sqrt{2}} = 1$$

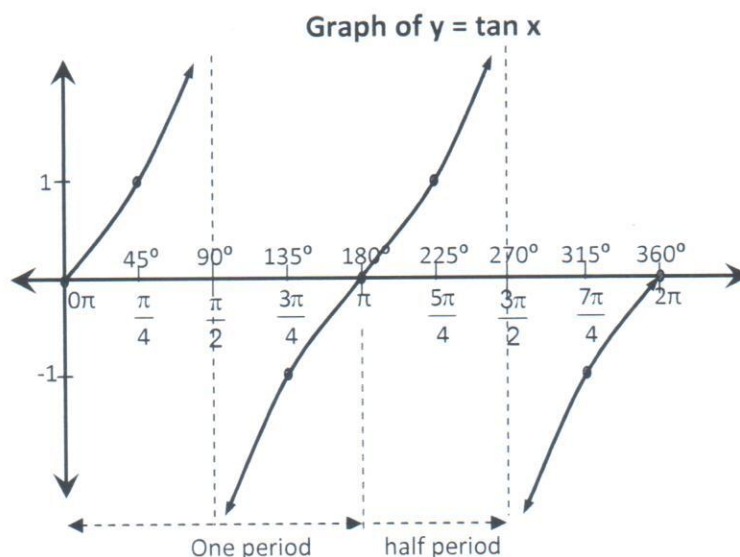
Therefore the value of tangent at $\frac{\pi}{4}$ is 1. These points have been added to the graph below.





Next, calculate the value of tangent for $\frac{3\pi}{4}$. Consulting the Common trigonometric Angle Measurements, the tangent of $\frac{3\pi}{4}$ is $\frac{\sqrt{2}}{2}$ (sine) divided by $-\frac{\sqrt{2}}{2}$ (cosine). This simplifies the tangent value of -1 .

Now, draw the tangent function graph so that the line approaches the asymptote without touching or crossing it. The graph below shows the completed graph of one and a half periods of the tangent function.



The period of the basic tangent function is π , and the graph will repeat from π to 2π .

The Form $y = A \sin(Bx + C) + D$

The Form $y = A \sin(Bx + C) + D$ is the general form of the sine function. From this form of the sine function, the amplitude, horizontal phase and vertical shifts from the basic trigonometric forms can be determined.

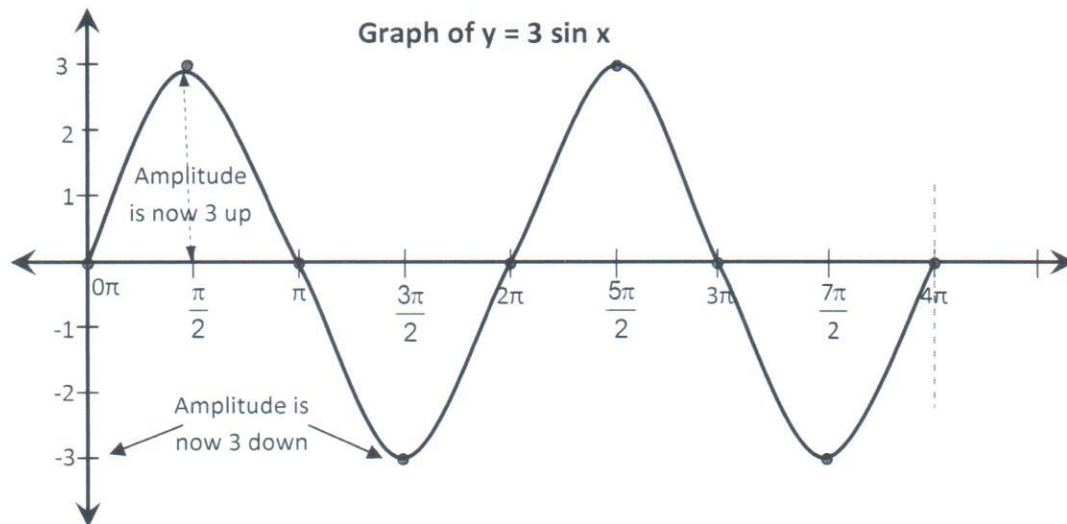
- A:** modifies the amplitude in the y direction above and below the centre line
- B:** influences the period and phase shift of the graph
- C:** influences the phase shift of the graph
- D:** shifts the centre line of the graph on the y-axis

Amplitude Shifts of Trigonometric functions

The basic graphs illustrate the trigonometric functions when the value of **A** is 1. This **A = 1** is used as an amplitude value of 1. If the value of A is not 1, then the absolute value of A is the new amplitude of the function. Any number $|A|$ greater than 1 will vertically stretch the graph (increase the amplitude) while a number $|A|$ smaller than 1 will compress the graph closer to the x-axis.

**Example 1** Graph $y = 3 \sin x$.

Solution: The graph of $y = 3 \sin x$ is the same as the graph of $y = \sin x$ except the minimum and maximum of the graph has been increased to -3 and 3 respectively from -1 and 1.

**Horizontal Shift of Trigonometric Function**

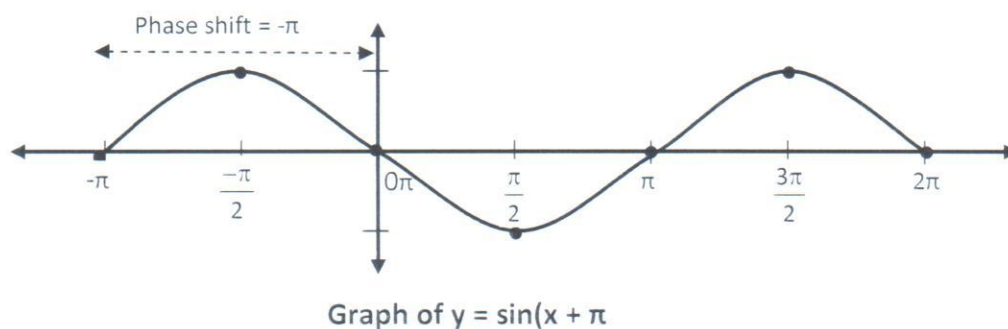
A horizontal shift is when the entire graph shifts left or right along the x-axis. This is shown symbolically as $y = \sin(Bx - C)$. Note the minus sign in the formula. To find the phase shift (or the amount the graph shifted) divide C by B.

For examples:

The phase shift of $y = \cos(2x - \pi)$ can be found by dividing π (C) by (B), and the answer is $\frac{\pi}{2}$.

The phase shift of $y = \sin(-2x - \pi)$ can be found by dividing $-\pi$ (C) by -2 (B), and the answer is $\frac{\pi}{2}$.

Note: Be careful when dealing with the signs. A positive sign takes the place of the double negative signs in the form $y = \sin(x + \pi)$. The C is negative because this example is also written as $y = \sin(x - (-\pi))$, which produces the negative π phase shift (see graph below).





Period Compression or Expansion of Trigonometric Function

The value of B also influences the period, or length of one cycle of trigonometric functions. The period of the basic sine and cosine functions is 2π while the period of the basic tangent function is π .

The period equation for sine and cosine is: **Period** = $\frac{2\pi}{|B|}$

The period equation for tangent is: **Period** = $\frac{\pi}{|B|}$

Period compression occurs if the absolute value of B is greater than 1. This means the function oscillates more frequently. Period expansion occurs if the absolute value of B is less than 1. This means the function oscillates more slowly.

The starting period of the graph is determined by the phase shift. To determine the key points for the new period, divide the period into 4 equal parts and add this part to successive x values beginning with the starting point.

Example 2 Graph $y = \sin(2x - \pi)$.

Solution: Identify A, B, C and D: $A = 1$, $B = 2$, $C = \pi$ and $D = 0$

The amplitude of A is 1.

The horizontal phase shift is $\frac{C}{B} = \frac{\pi}{2}$

The period changes to $\frac{2\pi}{|B|} = \frac{2\pi}{2} = \pi$

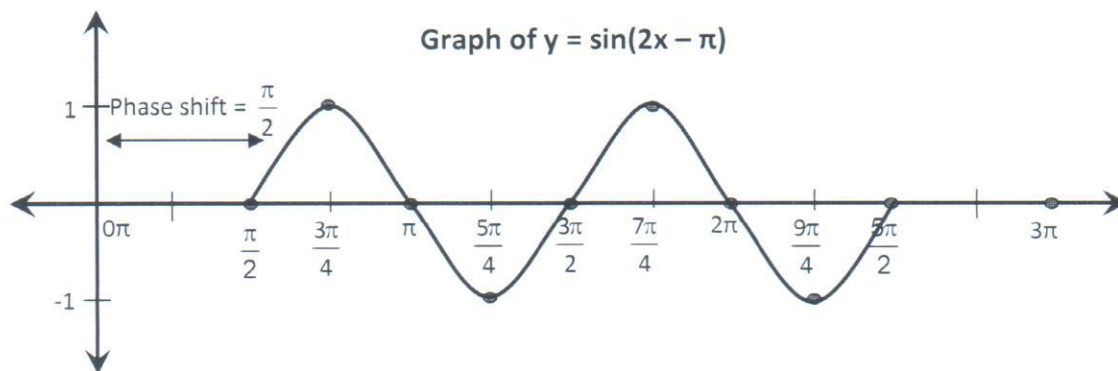
The starting point is $\frac{\pi}{2}$.

To find the 5 key points, divide π by 4 to obtain the value of the 4 equal parts: $\frac{\pi}{4}$

Then add $\frac{\pi}{4}$ to the starting point and each successive value of x to find the remaining four points. The 5 critical or key points are: $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π , $\frac{5\pi}{4}$ and $\frac{3\pi}{2}$.

Once the critical or key points are determined, evaluate the equation at these values and plot the resulting points.

The diagram on the next page shows the graph of the function.

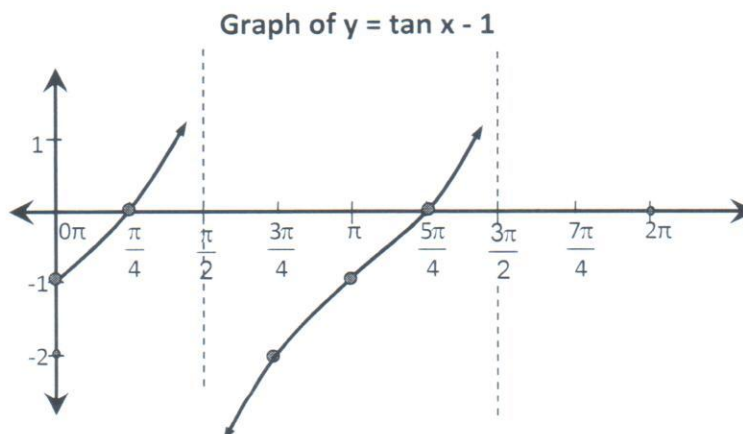


Vertical Shifts of Trigonometric Functions

A vertical shift occurs when the entire graph shifts up or down along the y-axis. This is known symbolically as $y = \sin x + D$. This is different from horizontal shifts because there are no parentheses around the D , and D is always a constant.

Example

The graph of $y = \tan x - 1$ would shift the entire graph down 1 unit.



In summary, it will make trigonometric functions easier to graph by using the following guidelines.

- Recall the values of sine and cosine on the unit circle.
- Identify the amplitude shift value A , in $y = A \sin x$, $y = A \cos x$, etc.
- Identify the horizontal shift value $\frac{C}{B}$, in $y = \sin(Bx - C)$, $y = \cos(Bx - C)$, etc.
- The period compression expansion of the graph is determined by dividing the period of the basic function by the absolute value of B .
- Identify the vertical shift value D , $y = \sin x + D$, $y = \cos x + D$, etc.
- The phase shift (if any) is the starting point to graphing the function. Divide the period by 4 to determine the 5 points to graph on the x-axis.
- Use the 5 points to determine corresponding y-coordinate.



Now look at the following work examples.

Example 1

For the function $y = \frac{1}{4} \sin x + 1$, find the amplitude, horizontal shift, period, and vertical shift of the following and graph one period.

Solution: First, find the values of A, B, C and D. from $y = \frac{1}{4} \sin x + 1$

$$A = \frac{1}{4}, B = 1, C = 0, \text{ and } D = 1$$

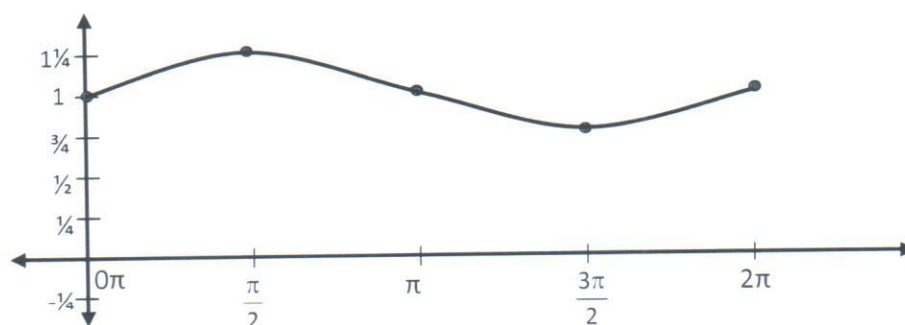
The amplitude shift is A. Hence, the amplitude is $\frac{1}{4}$.

$$\text{Horizontal Shift is } \frac{C}{B} = \frac{0}{1} = 0$$

$$\text{Period is } \frac{2\pi}{|1|} = 2\pi$$

$$\text{Vertical shift is } D = 1$$

Graph one period of the function $y = \frac{1}{4} \sin x + 1$



Example 2

For the function $y = \frac{1}{2} \cos(-x + \pi) - 1$, find the amplitude, horizontal shift, period, and vertical shift of the following and graph one period.

Solution: First find A, B, C and D from $y = \frac{1}{2} \cos(-x + \pi) - 1$

$$A = \frac{1}{2}, B = -1, C = -\pi \text{ and } D = -1$$

$$\text{Amplitude shift is A. so } A = \frac{1}{2}$$

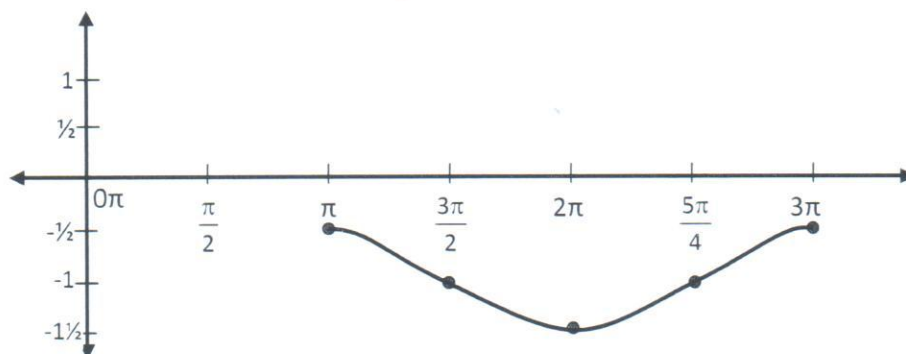
$$\text{Horizontal shift is } \frac{C}{B} = \frac{-\pi}{-1} = \pi$$



Period is $\frac{2\pi}{|-1|} = 2\pi$

Vertical shift is $D = -1$

The graph of one period of $y = \frac{1}{2} \cos(-x + \pi) - 1$



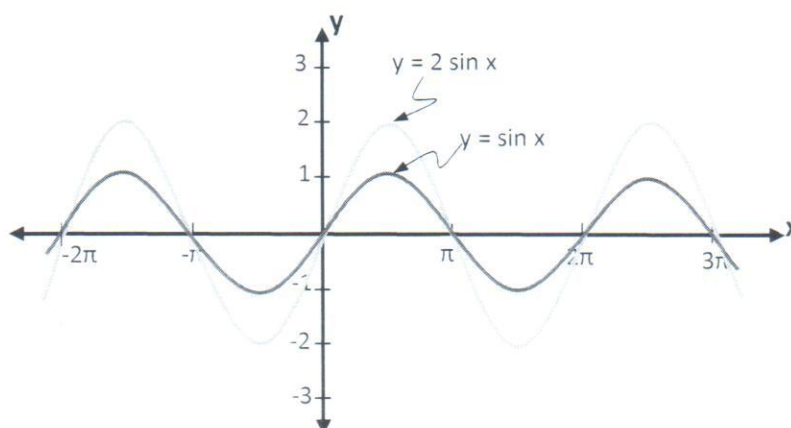
Graphs of Transformation of the Sine and Cosine

The basic trigonometric graphs can be transformed by horizontal or vertical translations, reflections or change of scale (stretch or compression). Note carefully the effect each transformation has on the equation of graph.

Example 1

Let us graph $y = 2 \sin x$.

Solution: We start with the graph of $y = \sin x$ and multiply the y -coordinate of each point by 2. This has the effect of stretching the graph vertically up or down by a factor of 2.

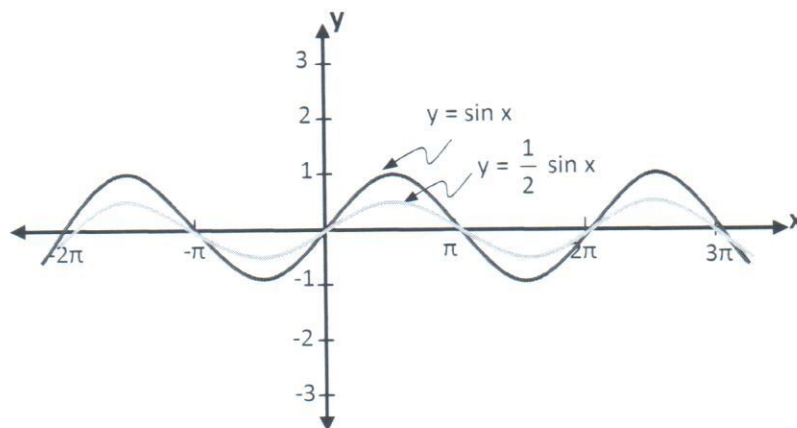




Example 2

Let us graph $y = \frac{1}{2} \sin x$.

Solution: We start with the graph of $y = \sin x$ and multiply the y -coordinate of each point by $\frac{1}{2}$. This has the effect of shrinking the graph vertically up or down by a factor of $\frac{1}{2}$.



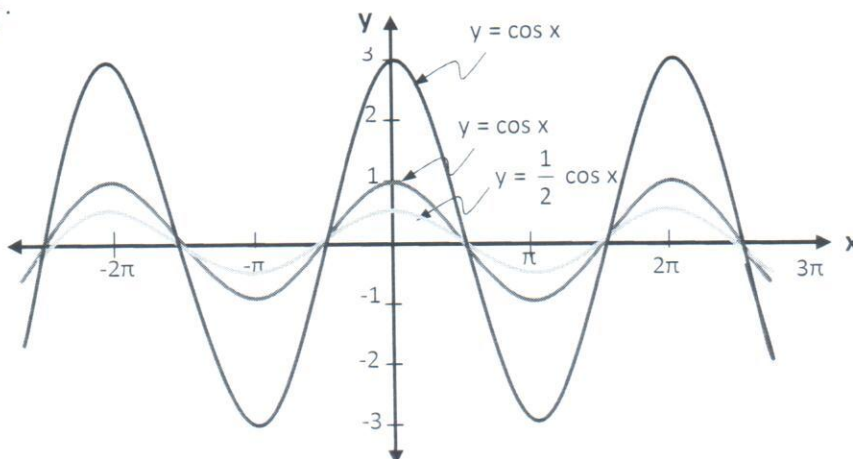
Example 3

Let us graph the function $y = 3 \cos x$ and $y = \frac{1}{2} \cos x$

Solution:

Solution: We start with the graph of $y = \cos x$ and multiply the y -coordinate of each point by 3. This has the effect of stretching the graph vertically up or down by a factor of 3.

We start with the graph of $y = \cos x$ and multiply the y -coordinate of each point by $\frac{1}{2}$. This has the effect of stretching the graph vertically up or down by a factor of $\frac{1}{2}$.





In general, for the functions

$$y = a \sin x \text{ and } y = a \cos x$$

a is called the **amplitude** and is the largest value these functions attain.

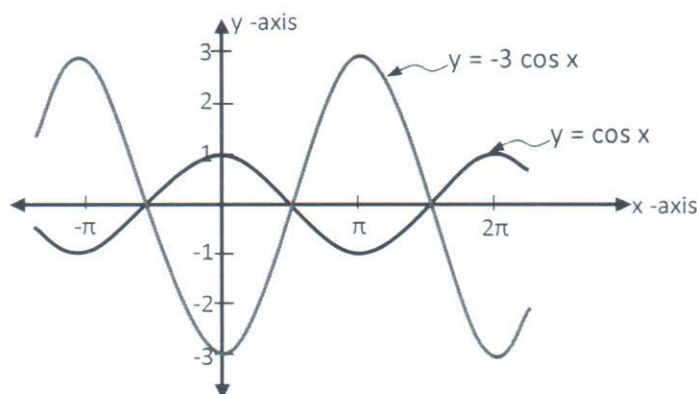
Here are other examples.

Example 4

Find the amplitude of $y = -3 \cos x$ and sketch the graph.

Solution: The amplitude is $|-3| = 3$ so the largest value the graph attains is 3 and the smallest value is -3.

To sketch graph the function, we start with the graph of $y = \cos x$. stretch the graph vertically by a factor of 3 and reflect in the x-axis, arriving at the graph shown below.



Example 5

Find the amplitude and period of $y = -2 \sin \frac{1}{2}x$.

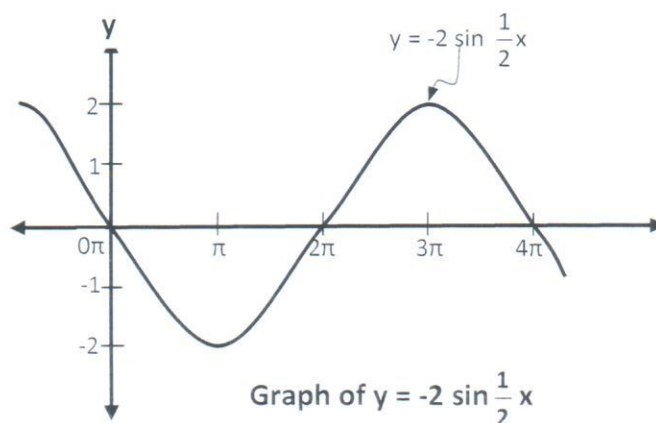
Solution:

We get the amplitude and period of the function $y = -2 \sin \frac{1}{2}x$.

The amplitude is $|a| = |-2| = 2$

The period is $\frac{2\pi}{\frac{1}{2}} = 2\pi(2) = 4\pi$

The graph is shown at the right.

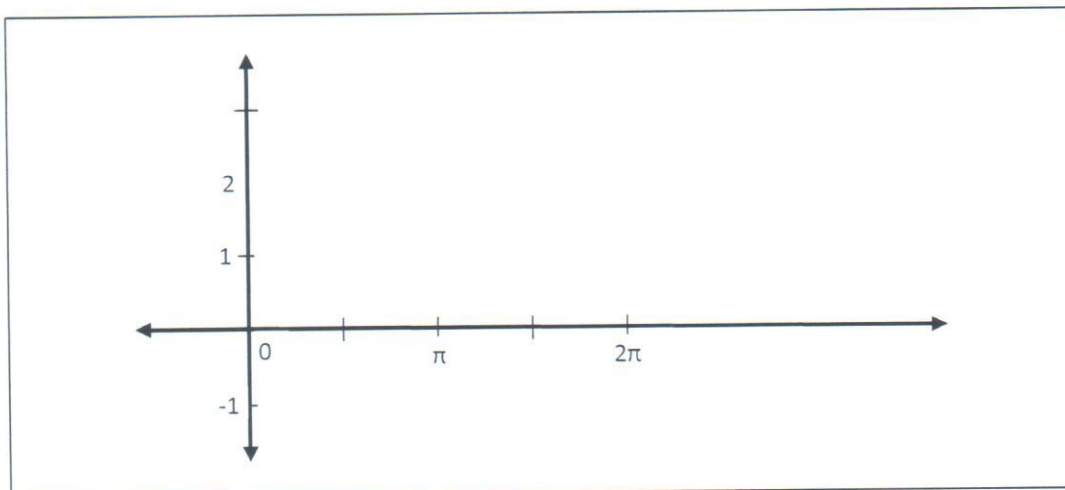


**LEARNING ACTIVITY 12.2.1.4.3**

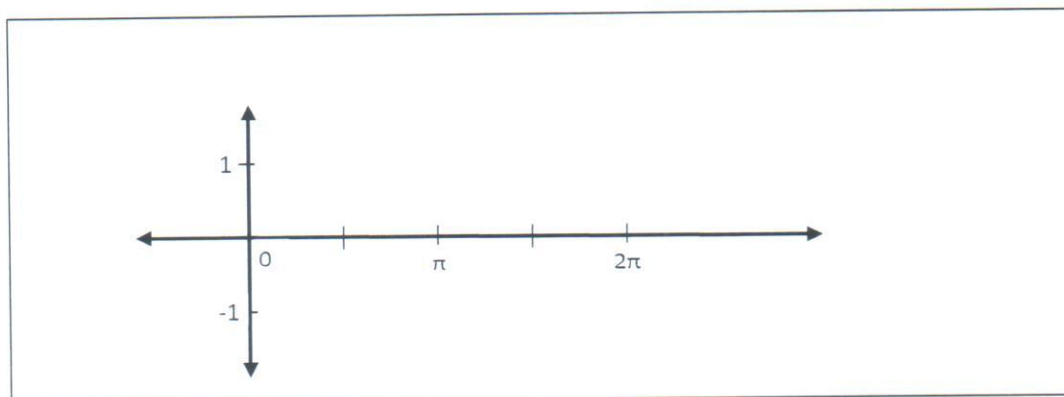
20 minutes

A. Sketch the graph of each function.

1. $f(x) = 2 + \cos x$



2. $g(x) = -\cos x$



B. Find the amplitude (A), the period and the phase shift (B) of each of the following function.

1. $y = 2 \cos 3x$

Amplitude =

Period =

Phase shift =



2. $y = -5 \cos \frac{1}{3}x$

Amplitude =

Period =

Phase shift =

3. $y = -2 \sin 4(x - 1.3)$

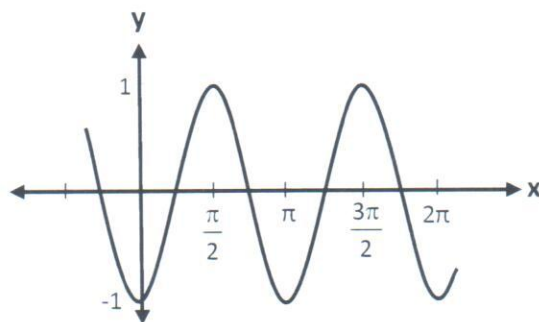
Amplitude =

Period =

Phase shift =

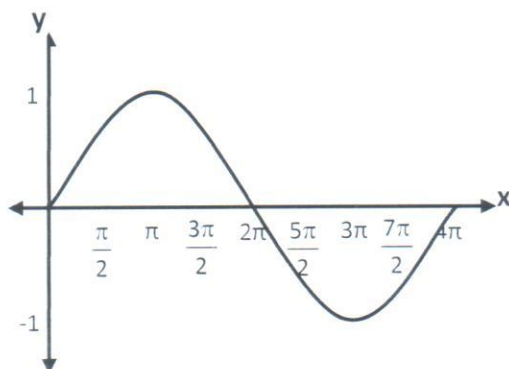
C. Sketch the graph of the following:

1. $y = -\cos 2x$



2. On an axes from 0 to 4π , graph of $y = \sin\left(\frac{1}{2}x\right)$.

Where on the graph does the graph reach its maximum value(s)?



The graph reaches its maximum in this interval at π .



12.2.1.5: The Graphs of Reciprocal Functions

We have covered the graphs of sine, cosine and tangent functions. In this section, we will look at the graphs of their reciprocal functions.

Sine, cosine, and tangent each have a **reciprocal** function. Reciprocal, reciprocity—think of flipping things over, like hamburgers on a grill, pancakes on a griddle, eggs over easy. We already know that regular numbers have reciprocals (2 and $\frac{1}{2}$ are reciprocals, for example), but we can also flip our trigonometric functions.

The three reciprocal trigonometric functions are:

1. **Cosecant** is the reciprocal of sine. Its abbreviation is **csc**. To determine csc, just flip sin over.
2. **Secant** is the reciprocal of cosine. Its abbreviation is **sec**. To determine sec, just flip cos over.
3. **Cotangent** is the reciprocal of tangent. Its abbreviation is **cot**. To determine cot, just flip tan over.

The table below shows

Reciprocal ratios	
$\sin \theta = \frac{1}{\csc \theta}$	$\csc \theta = \frac{1}{\sin \theta}$
$\cos \theta = \frac{1}{\sec \theta}$	$\sec \theta = \frac{1}{\cos \theta}$
$\tan \theta = \frac{1}{\cot \theta}$	$\cot \theta = \frac{1}{\tan \theta}$

12.2.1.5.1: The Graph of the Cosecant, Secant and Cotangent Functions

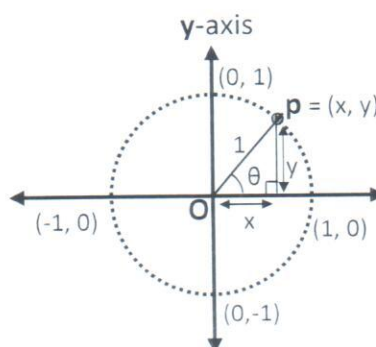
The graph of the cosecant, secant and cotangent functions can be obtained from the graphs of the sine and cosine functions using the reciprocal relations.

For example, in the unit circle, at a given value of x , the y -coordinate of the $\sec \theta$ is the reciprocal of the y -coordinate of $\cos \theta$.

$$\sec \theta = \frac{1}{\cos \theta} = \frac{1}{x}, x \neq 0$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{1}{y}, y \neq 0$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{x}{y}, y \neq 0$$





When $\cos \theta = 0$, the reciprocal does not exist. Near such values of θ , the secant function is similar to the tangent function.

In other words, the graphs of

$$\tan x = \frac{\sin x}{\cos x} \text{ and } \sec x = \frac{1}{\cos x}$$

have vertical asymptotes at $\theta = \frac{\pi}{2} + n\pi$, where n is an integer, and the cosine is zero at these x -values. Similarly,

$$\cot x = \frac{\cos x}{\sin x} \text{ and } \csc x = \frac{1}{\sin x}$$

have vertical asymptote where $\sin \theta = 0$, that is, at $\theta = n\pi$.

To sketch the graph of a secant and cosecant functions, you should first make a sketch of their reciprocal functions.

Example 1

Sketch the graph of $y = \csc x$.

Solution: Cosecant function is the reciprocal of sine function.

The sketch of $y = \sin x$ can be used to draw graph $y = \csc x$.

Since $\sin x = 0$ when $x = 0, \pi, 2\pi$.

Therefore, $y = \csc x \Rightarrow y = \frac{1}{\sin x}$ is undefined when $x = 0, \pi, 2\pi$

so, the vertical asymptotes are: $x = 0, x = \pi, x = 2\pi$

Clearly,

$$\text{As } x \rightarrow 0, y = -\infty$$

$$\text{As } x \rightarrow 0, y = -\infty$$

$$\text{As } x \rightarrow 0, y = \infty$$

$$\text{As } x \rightarrow 0, y = \infty$$

Also

$$\text{When } x = \frac{\pi}{2}, y = 1$$

$$\text{when } x = -\frac{\pi}{2}, y = -1$$

$$\text{When } x = \frac{3\pi}{2}, y = -1$$

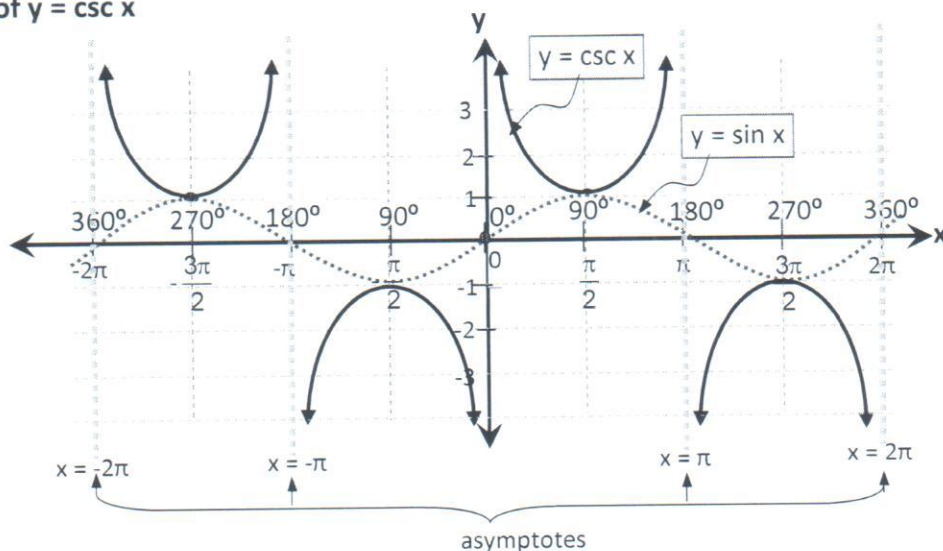
$$\text{When } x = -\frac{3\pi}{2}, y = 1$$

So the points $(\frac{\pi}{2}, 1), (\frac{3\pi}{2}, -1), (-\frac{\pi}{2}, -1), (-\frac{3\pi}{2}, 1)$ lie on the graph.



Using these information we can obtain the graph shown below.

The graph of $y = \csc x$



Looking at the graph, we can see that:

- there are vertical asymptotes. The asymptote that occurs at π repeats every π units.
- the function is periodic. The period is 2π .
- the function has no amplitude. The graphs go on forever in vertical directions.
- the maximum values of $y = \sin x$ are minimum values of the positive sections of $y = \csc x$. The minimum values of $\sin x$ are the maximum values of the negative sections of $y = \csc x$.
- the x-intercepts of the sine function become the asymptotes of the cosecant function.

Note: the U shapes of the cosecant graph are tangent to its reciprocal function, sine, at sine's maximum and minimum locations.

Example 2

Sketch the graph of $y = \sec x$.

Solution: Secant function is the reciprocal of cosine function.

The graph of $y = \cos x$ can be used to draw graph $y = \sec x$.

Therefore, $y = \sec x \Rightarrow y = \frac{1}{\cos x}$ is undefined when $x = \frac{\pi}{2}, \frac{3\pi}{2}$.

so, the vertical asymptotes are: $x = \frac{\pi}{2}, x = \frac{3\pi}{2}$

Clearly,

$$\text{As } x \rightarrow \frac{\pi}{2}, y \rightarrow -\infty \quad \text{As } x \rightarrow \frac{3\pi}{2}, y \rightarrow -\infty$$

$$\text{As } x \rightarrow -\frac{\pi}{2}, y \rightarrow \infty \quad \text{As } x \rightarrow -\frac{3\pi}{2}, y \rightarrow \infty$$



Also

When $x = 0$, $y = 1$

When $x = \pi$, $y = -1$

When $x = 2\pi$, $y = 1$

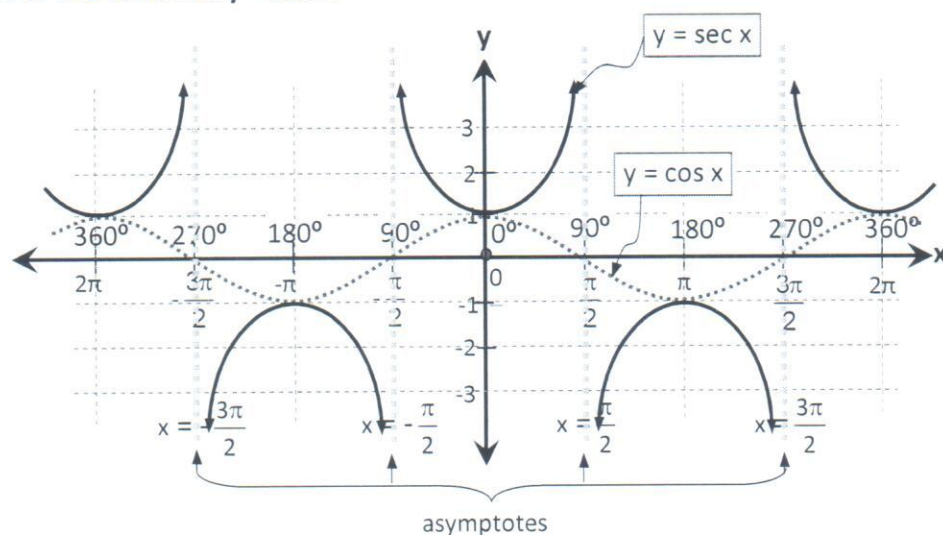
When $x = -\pi$, $y = -1$

When $x = -2\pi$, $y = 1$

So the points $(0, 1)$, $(\pi, -1)$, $(2\pi, 1)$, $(-\pi, -1)$ lie on the graph.

Now we use these information to obtain the graph below.

The graph of the function $y = \sec x$



Looking at the graph, we can see that:

- there are vertical asymptotes. The asymptote that occurs at $\frac{\pi}{2}$ repeats every π units.
- the function is periodic. The period is 2π .
- the function has no amplitude. The graphs go on forever in vertical directions.
- the maximum values of $y = \cos x$ are minimum values of the positive sections of $y = \sec x$. The minimum values of $\cos x$ are the maximum values of the negative sections of $y = \sec x$.
- the x-intercepts of the cosine function become the asymptotes of the secant function.

Note: the U shapes of the cosecant graph are tangent to its reciprocal function, cosine, at cosine's maximum and minimum locations.



Example 3

Sketch the graph of $y = \cot x$.

Solution:

Since cotangent function is the reciprocal of tangent function.

Then graph of $y = \tan x$ can be used to draw the graph of $y = \cot x$.

Since, $y = \cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$, $\sin x$ is undefined when $x = 0, \pi$ and 2π

Therefore, the vertical asymptotes are $x = 0, x = \pi$ and $x = 2\pi$

Clearly

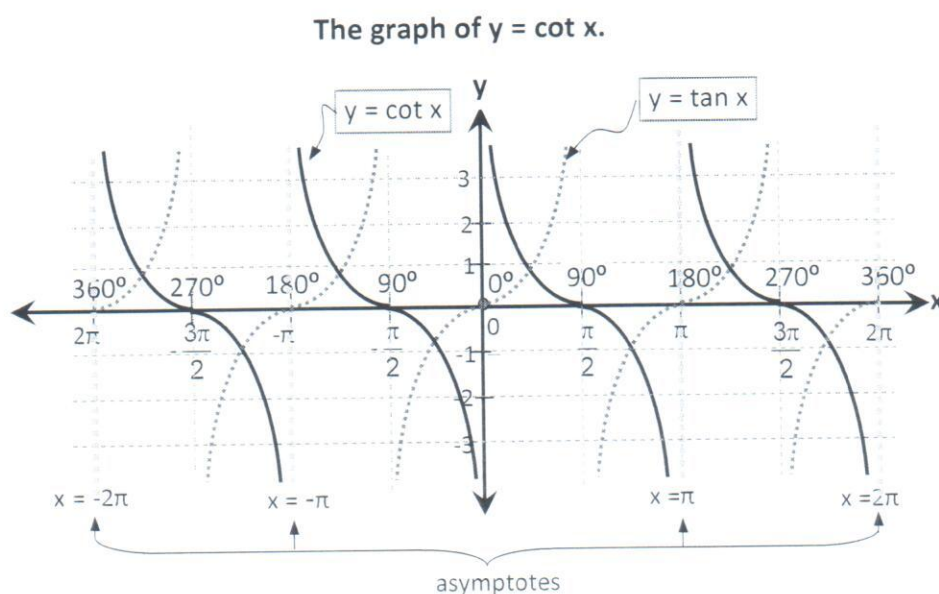
- As $x \rightarrow 0, y = \infty$
- As $x \rightarrow \pi, y = \infty$
- As $x \rightarrow -\pi, y = -\infty$
- As $x \rightarrow 2\pi, y = \infty$
- As $x \rightarrow -2\pi, y = -\infty$

Also

When $x = \frac{\pi}{4}, y = 1$	When $x = -\frac{\pi}{4}, y = -1$
When $x = \frac{3\pi}{4}, y = -1$	When $x = -\frac{3\pi}{4}, y = 1$
When $x = \frac{5\pi}{4}, y = 1$	When $x = -\frac{5\pi}{4}, y = -1$
When $x = \frac{7\pi}{4}, y = -1$	When $x = -\frac{7\pi}{4}, y = 1$

So, points $(\frac{\pi}{4}, 1), (\frac{3\pi}{4}, -1), (\frac{5\pi}{4}, 1), (\frac{7\pi}{4}, -1), (-\frac{\pi}{4}, -1), (-\frac{3\pi}{4}, 1), (-\frac{5\pi}{4}, -1), (-\frac{7\pi}{4}, 1)$ lie on the graph.

Using these information, we obtain the graph shown below.





Studying the graph, we can see that:

- one cycle occurs between 0 and π .
- There are vertical asymptotes at each end of the cycle. The asymptote that occurs at π repeats every π units.
- The period of the function is π
- There is no amplitude, the graphs go on forever in vertical directions
- The x-intercepts of the graph of $y = \tan x$ are the asymptotes of the graph of $y = \cot x$.
- The asymptotes of the graph of $y = \tan x$ are the x-intercepts of the graph of $y = \cot x$.
- The graphs of $y = \tan x$ and $y = \cot x$ “face” in opposite directions.

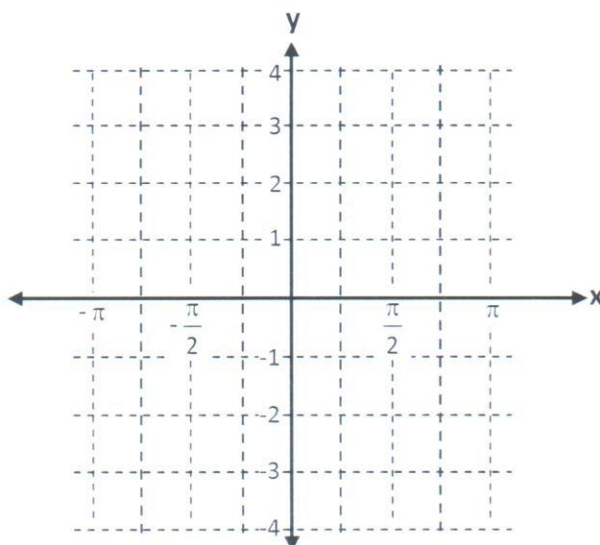


LEARNING ACTIVITY 12.2.1.5



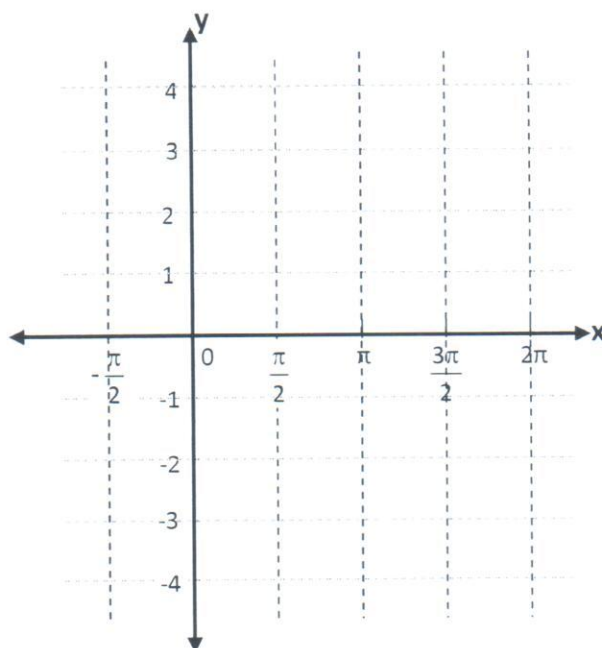
20 minutes

1. Sketch the graph of $y = \sec 2x$. use the grid below.

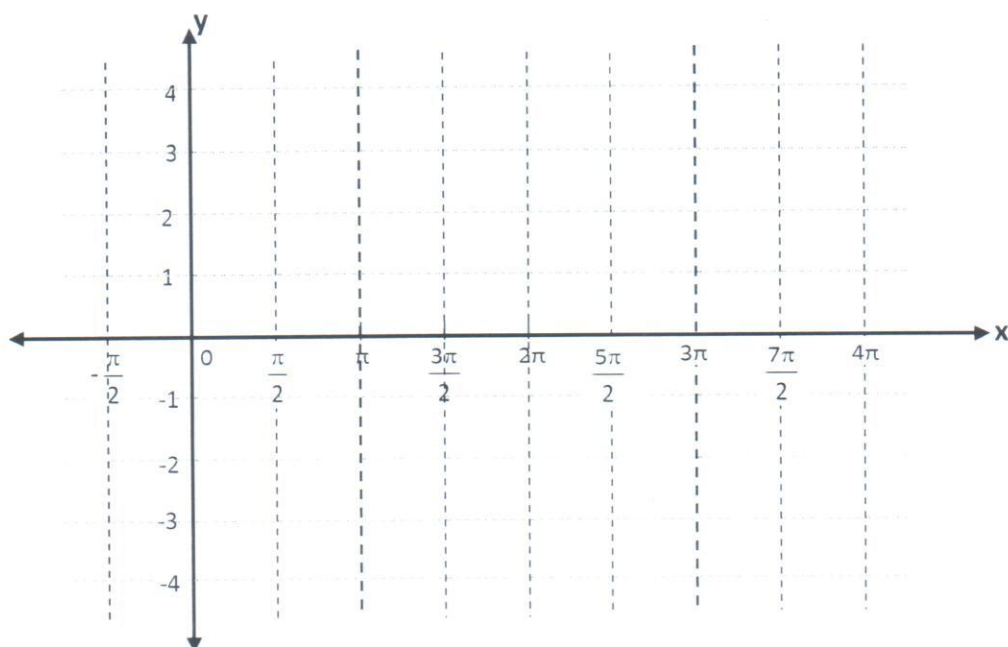




2. Sketch the graph of $y = 2 \sec x$ in the domain $\{0 \leq x \leq 2\pi\}$. Use the grid below.



3. Sketch the graph of $y = 2 \tan \frac{x}{2}$ in the domain $\{0 \leq x \leq 4\pi\}$. Use the grid below.





12.2.1.6: The Sine and Cosine Rules

The sine and cosine rules are used to solve scalene triangles in which there are no right angles. Right-angled triangles and isosceles triangles are easier to solve using Pythagoras' theorem and the trigonometric ratios sine, cosine and tangent.

When trying to solve any triangle it is necessary to draw a clear diagram first. Label carefully with the given information. Mark the value or values you have to calculate.

To solve a triangle is to find the lengths of each side and all its angles. The **sine rule** is used when you are given either two angles and one side, or two sides and a non-included angle. The **cosine rule** is used when you are given either three sides or two sides and the included angle.

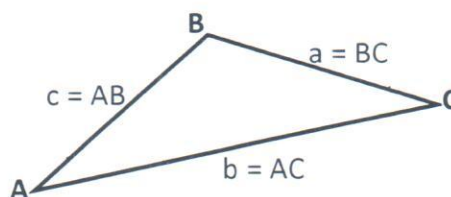
12.3.1.6.1: The Sine Rule

For any triangle ABC, the sine rule can be written as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Finding the Sides

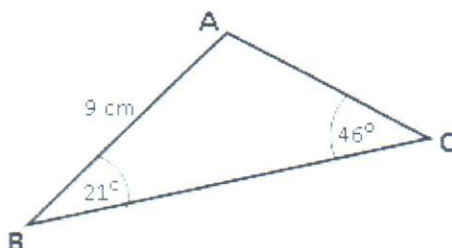
If you need to find the length of a side, you need to use the sine rule in this form:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

You need to know the length of another sides and the size of any two angles.

Example 1

In triangle ABC shown below, $\angle B = 21^\circ$, $\angle C = 46^\circ$ and $AB = 9$ cm. Solve this triangle.





Solution:

We are given two angles, $\angle B = 21^\circ$, $\angle C = 46^\circ$ and one side, $AB = 9$ cm and so the sine rule can be used. Furthermore, since the angles in a triangle sum up to 180° , then $\angle A$ must be 113° . we know that $c = AB = 9$ cm. Using the sine rule we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Leftrightarrow \frac{a}{\sin 113^\circ} = \frac{b}{\sin 21^\circ} = \frac{9 \text{ cm}}{\sin 46^\circ}$$

So, to find **b**; use $\frac{b}{\sin B} = \frac{c}{\sin C} \Leftrightarrow \frac{b}{\sin 21^\circ} = \frac{9 \text{ cm}}{\sin 46^\circ}$

From which,

$$b = \sin 21^\circ \left(\frac{9 \text{ cm}}{\sin 46^\circ} \right)$$

$$b = 0.3583 \left(\frac{9 \text{ cm}}{0.7194} \right)$$

$$b = \frac{3.2247}{0.7194} \text{ cm}$$

$$b = 4.484 \text{ cm or } 4.5 \text{ cm}$$

To find **a**, use $\frac{a}{\sin A} = \frac{c}{\sin C} \Leftrightarrow \frac{a}{\sin 113^\circ} = \frac{9 \text{ cm}}{\sin 46^\circ}$

$$a = \sin 113^\circ \left(\frac{9 \text{ cm}}{\sin 46^\circ} \right)$$

$$a = 0.9205 \left(\frac{9 \text{ cm}}{0.7194} \right)$$

$$a = \frac{8.2845}{0.7194} \text{ cm}$$

$$a = 11.517 \text{ cm or } 11.5 \text{ cm}$$

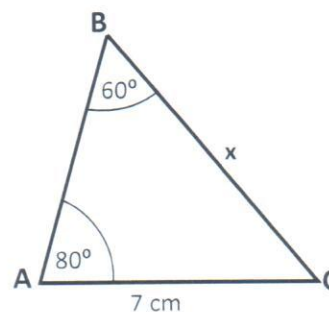
Example 2

Work out the length **x** in the diagram.

Solution:

You will only need two parts of the sine rule.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$



Fill in the known values and the unknown length.

$$\frac{x}{\sin 80^\circ} = \frac{7 \text{ cm}}{\sin 60^\circ}$$

$$x = \frac{7 \text{ cm}}{\sin 60^\circ} (\sin 80^\circ) = \frac{7 \text{ cm} (0.9848)}{0.8660} = \frac{6.8936}{0.8660} \text{ cm} = 7.96 \text{ cm}$$

**Finding angles**

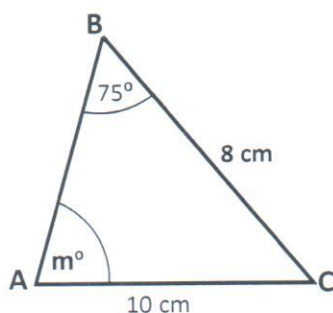
If you need to find the size of an angle, you need to use the sine rule in the form where the angles are on the top.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

You still need at least a side and its opposite angle.

Example 3

Work out angle m° in the diagram below.



Solution:

Use the sine rule formula $\frac{\sin A}{a} = \frac{\sin B}{b}$.

Fill in the values you know and the unknown angle. Remember that each fraction in the sine rule formula should contain a side and its opposite angle.

Hence we have,
$$\frac{\sin(m^\circ)}{8 \text{ cm}} = \frac{\sin 75^\circ}{10 \text{ cm}}$$

Solve for the value of m .
$$\sin(m^\circ) = \frac{(8 \text{ cm}) \sin 75^\circ}{10 \text{ cm}}$$

$$\sin(m^\circ) = \frac{(8 \text{ cm})(0.9659)}{10 \text{ cm}}$$

$$\sin(m^\circ) = \frac{7.7272 \text{ cm}}{10 \text{ cm}}$$

$$\sin(m^\circ) = 0.773$$

Use the inverse sine function $\sin^{-1}$ function to find the angle.

$$m^\circ = \sin^{-1}(0.773)$$

$$m = 50.62405590\dots$$

$$m = 50.6^\circ \text{ (correct to 3 significant figures)}$$

To find the value of m in degrees using your calculator,

Press calculator KEYS Display

0.773 INV $\sin^{-1}$ 50.62405590...

**LEARNING ACTIVITY 12.2.1.6.1**

20 minutes

In the problems below, **a**, **b** and **c** refer to the lengths of sides of the triangle and **A**, **B** and **C** refer to the corresponding opposite angles.

1. Given: $a = 3$, $b = 4$, $A = 40^\circ$.

Find: **B**

2. Given: $b = 6$ cm, $B = 77^\circ$ and $C = 34^\circ$.

Find: a. **c**

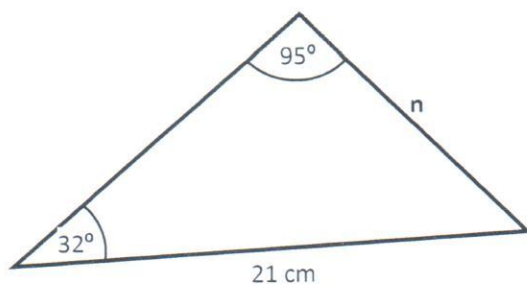
b. **A**

c. **a**

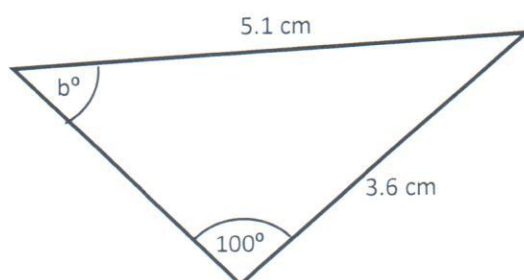


3. Work out the answer to each question below.

a. Find the missing side in the triangles below.



b. find the missing angle in the triangle below.



12.2.1.6.2: The Cosine Rule

For any triangle ABC, the cosine rule can be written as

$$a^2 = b^2 + c^2 - 2bc \cos A$$

This formula can also be given with **b** and **c** as the subject.

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

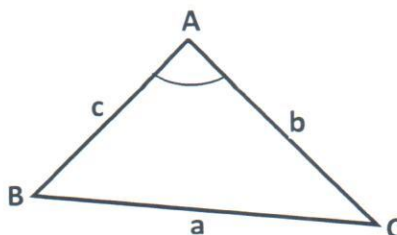


Finding sides

When we know the two sides and the angle between them (SAS), we can find the third side using this rule.

Given: Two sides c and b
Included angle $\angle A$

Unknown: side a

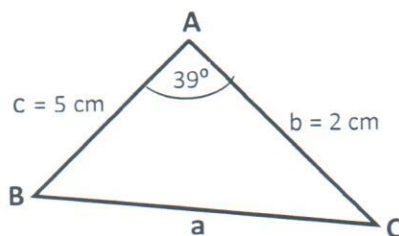


The given angle may be acute or obtuse angle. If it is obtuse, then the cosine is negative.

Now look at the examples.

Example 1

In triangle **ABC** below, $b = 2$ cm, $c = 5$ cm and $\angle A = 39^\circ$. Find the length of a .



Solution: We are given two sides b and c and their included angle A .

To find a , use the cosine rule.
$$a^2 = b^2 + c^2 - 2bc \cos A$$

Fill in the known values and the unknown length.

From which, we have
$$a^2 = 2^2 + 5^2 - 2(2)(5) \cos 39^\circ$$

And so
$$a^2 = 4 + 25 - 20(0.7771)$$

$$a^2 = 29 - 15.542$$

$$a^2 = 13.458$$

$$a = \sqrt{13.458}$$

$$a = 3.67 \text{ cm}$$

Note: In problems where no diagram is given, it does not matter which way around you put the two given sides, b and c , it will work both ways.

**Example 2**

Find the missing side marked x of the triangle at the right.

Solution:

Use the cosine rule formula: $a^2 = b^2 + c^2 - 2bc \cos A$

Fill in the values given and the unknown length .

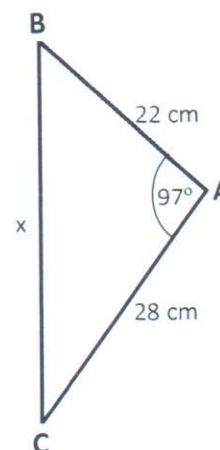
$$x^2 = 22^2 + 28^2 - 2(22)(28)\cos 97^\circ$$

$$x^2 = 484 + 784 - (1232)(0.1219)$$

$$x^2 = 1268 - 150.1808$$

$$x^2 = 1117.8192$$

$$x = 33.4 \quad \text{correct to 3 significant figures}$$

**Finding Angles**

The cosine rule can be written as

for $\angle A$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

for $\angle B$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

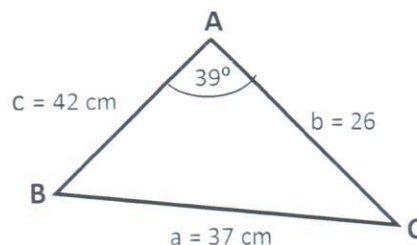
for $\angle C$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

When you know all the three angles you can use these forms of the cosine rule to find an angle in a triangle.

Example 3

In the triangle ABC, $AB = 42$ cm, $BC = 37$ cm and $AC = 26$ cm. Solve this triangle.



Solution:

We are given three sides of the triangle and so the cosine rule can be used.

Writing $a = 37$ cm, $b = 26$ cm and $c = 42$ cm in the formula, we have



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

From which $\cos A = \frac{26^2 + 42^2 - 37^2}{2(26)(42)}$

$$\cos A = \frac{676 + 1764 - 1369}{2184}$$

$$\cos A = \frac{1071}{2184}$$

$$\cos A = 0.4904$$

And so

$$A = \cos^{-1} 0.4904$$

$$A = 60.63312419\dots$$

$$A = 60.63^\circ \text{ (correct to 3 significant figures)}$$

To find the value of A in degrees using your calculator.

Press calculator KEYS Display

0.4904 INV $\cos^{-1}$ 60.63312419...

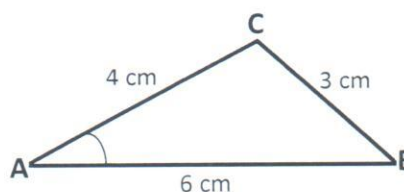
Here is another example.

Example 4

In $\triangle ABC$, sides $\overline{AB} = 6$ cm, $\overline{AC} = 4$ cm and $\overline{BC} = 3$ cm. calculate the size of angle A correct to one decimal point.

Solution:

Use the cosine rule



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{4^2 + 6^2 - 3^2}{2(4)(6)}$$

$$\cos A = \frac{16 + 36 - 9}{48}$$

$$\cos A = \frac{43}{48}$$

$$\cos A = 0.8958$$

$$A = 26.4^\circ$$

To find the value of A in degrees using your calculator.

Press calculator KEYS Display

0.8958 INV $\cos^{-1}$ 26.38862713...

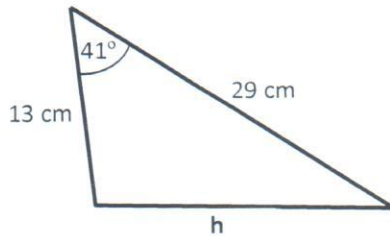
**LEARNING ACTIVITY 12.2.1.6.2**

20 minutes

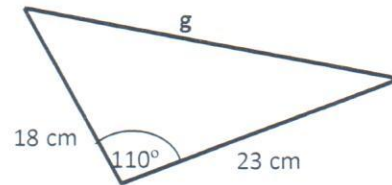
Work out the answer to each question using the cosine rule.

1. Find the missing side in the diagram below.

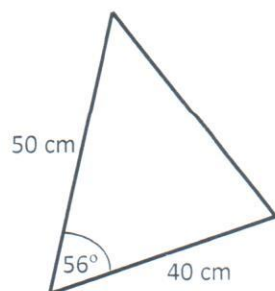
a.



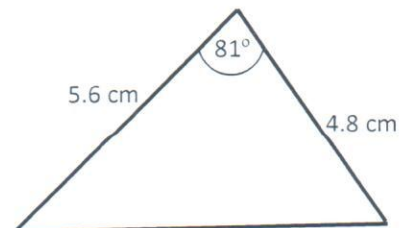
b.



c.



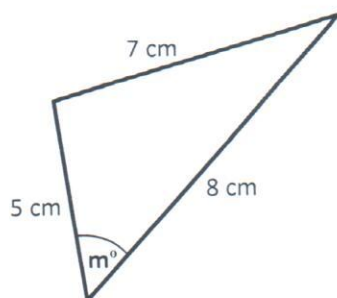
d.



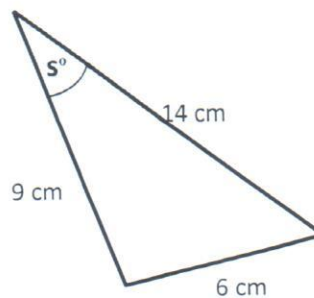


2. Find the missing angle in the diagram below.

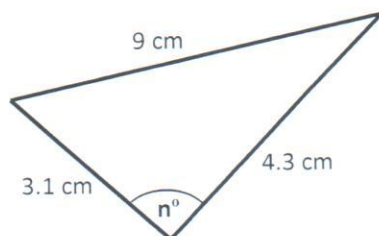
a.



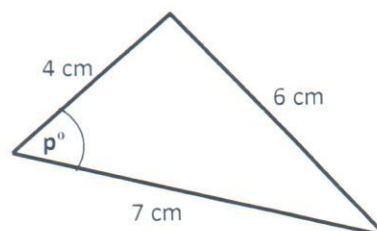
b.



c.



d.



Now do the summative task on the next pages.



SUMMATIVE TASK 12.2.1

Answer each of the following questions.

1. Find the radian measure for a central angle of:

a. 36°

b. -45°

c. 180°

d. 217°

2. Find the degree measure which corresponds to measure of:

a. $\frac{3\pi}{2}$

b. $\frac{23\pi}{12}$

c. $-\frac{7\pi}{6}$

d. $\frac{19\pi}{6}$



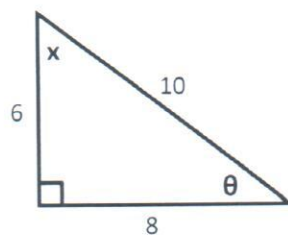
3. For this right triangle, give the following ratios:

a. $\tan x$

b. $\frac{\sin x}{\cos x}$

c. $\cos \theta$

d. $\sin \theta \cdot \cos \theta$



4. Find the exact value of each trigonometric function. Use the table on page 30.

a. $\tan \frac{2\pi}{3}$

b. $\sec -\frac{3\pi}{4}$

c. $\cos \frac{11\pi}{3}$

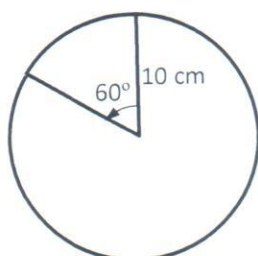
d. $\sec 0$

e. $\csc -\frac{\pi}{2}$

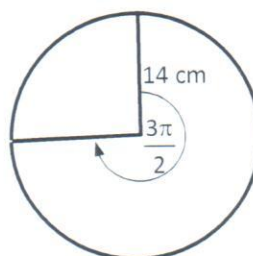
e. $\cot \frac{5\pi}{3}$

5. Find the area of each sector.

a.



b.



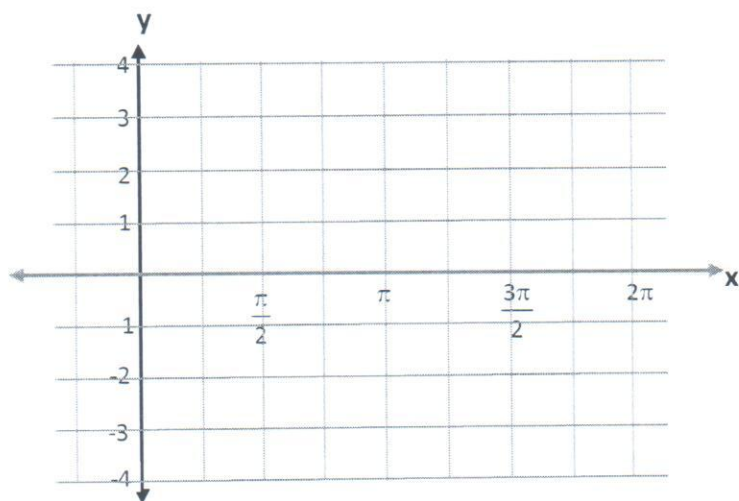


6. Sketch each graph and find its period and amplitude.

a. $y = \sin 3x$

Amplitude =

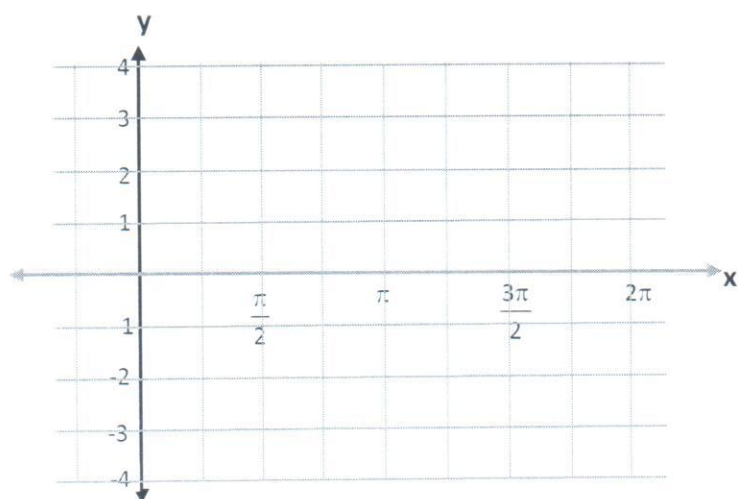
Period =



b. $y = 2 \cos 4x$

Amplitude =

Period =





12.2.2: TRIGONOMETRIC IDENTITIES AND EQUATIONS

In this section, we will define trigonometric equations, their solutions as well as the methods for solving trigonometric equations.

Trigonometric equations are equations involving trigonometric functions of unknown angles.

There are two types of trigonometric equations:

1. **Identical equations** or **Identities** are those equations satisfied by all values of the unknown angles for which the functions are derived.

Examples: $\tan x \cot x = 1$ is an identity since it is satisfied by every value of x for which $\tan x$ is defined.

2. **Conditional equations** or simply **equations** are those equations satisfied only by particular values of unknown angles.

Example: $\cos x = 0$ is a conditional equation since it is satisfied only by $x = \frac{\pi}{2}, \frac{3\pi}{2}$, etc.

12.2.2.1: Fundamental Trigonometric Identities

If a trigonometric equation is true for all values of the variable in the specified domain, then the equation is said to be a **Trigonometric Identities**.

For example, $\tan x = \frac{\sin x}{\cos x}$ ($\cos x \neq 0$) is a trigonometric identity, as it is true for all values of x in the specified domain (i.e. provided $\cos \neq 0$).

The basic trigonometric identities come in several varieties. These include the **reciprocal identities**, **ratio identities**, **Pythagorean identities**, **symmetric identities**, and **cofunction identities**. Each of these identities follows directly from the definition.

Reciprocal identities

The simplest and most basic trigonometric identities (equations of equivalence) are those involving the **reciprocals** of the trigonometry functions.

A reciprocal of a number is 1 divided by that number.

For example, the reciprocal of 2 is $\frac{1}{2}$. Another way to describe reciprocals is to point out that the product of a number and its reciprocal is 1. In case of 2 and its reciprocal $\frac{1}{2}$: $2(\frac{1}{2}) = 1$. The same goes for the trigonometric reciprocals.



Here's how the reciprocal identities are defined:

- The reciprocal of sine is cosecant:

$$\frac{1}{\sin \theta} = \csc \theta$$

- The reciprocal of cosine is secant:

$$\frac{1}{\cos \theta} = \sec \theta$$

- The reciprocal of tangent is cotangent:

$$\frac{1}{\tan \theta} = \cot \theta$$

- The reciprocal of cotangent is tangent:

$$\frac{1}{\cot \theta} = \tan \theta$$

- The reciprocal of secant is cosine:

$$\frac{1}{\sec \theta} = \cos \theta$$

- The reciprocal of cosecant is sine:

$$\frac{1}{\csc \theta} = \sin \theta$$

In real approach, when you multiply the reciprocals together, you get 1:

$$\sin \theta \cdot \csc \theta = 1$$

$$\cos \theta \cdot \sec \theta = 1$$

$$\tan \theta \cdot \cot \theta = 1$$

Note: The function cannot be equal to 0; the number 0 does not have a reciprocal. The reciprocal identity is a very useful one when you're solving trig equations. If you find a way to multiply each side of an equation by a function's reciprocal, you may be able to reduce some part of the equation to 1 and simplifying is always a good thing.

Ratio Identities

The **ratio identities** are also sometimes referred to as the **quotient identities**.

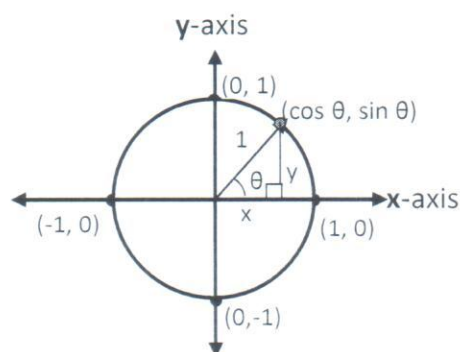
$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

Proof: From the **Unit Circle Definition**,

$$\text{we have } \tan \theta = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$

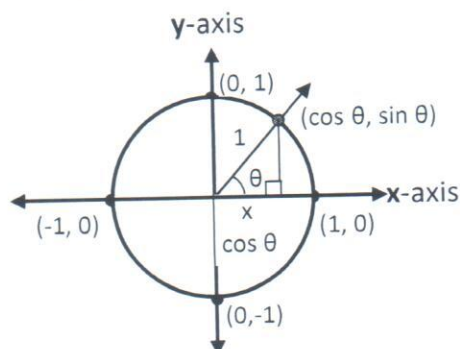
$$\text{Similarly, we have } \cot \theta = \frac{x}{y} = \frac{\cos \theta}{\sin \theta}$$





Pythagorean Identities

When studying the unit circle, it was observed that a point on the unit circle (the vertex of the right triangle) can be represented by the coordinates $(\cos \theta, \sin \theta)$.



Since the legs of the right triangle in the unit circle have the values of $\sin \theta$ and $\cos \theta$, the Pythagorean Theorem can be used to obtain $\sin^2 \theta + \cos^2 \theta = 1$.

$$\sin^2 \theta + \cos^2 \theta = 1$$

This well-known equation is called a **Pythagorean Identity**.

Using this first Pythagorean Identity, two other Pythagorean identities can be created.

Start with the first Pythagorean identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

Divide each term by $\cos^2 \theta$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

Remember, $\frac{\sin \theta}{\cos \theta} = \tan \theta$ and $\frac{1}{\cos \theta} = \sec \theta$

Reduce and substitute.

$$\tan^2 \theta + 1 = \sec^2 \theta$$

The second Pythagorean identities is

$$\tan^2 \theta + 1 = \sec^2 \theta$$

Now for the third equation:

Start with the first Pythagorean identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

Divide each term by $\sin^2 \theta$

$$\frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$

Remember, $\frac{\cos \theta}{\sin \theta} = \cot \theta$ and $\frac{1}{\sin \theta} = \csc \theta$

Reduce and substitute.

$$\cot^2 \theta + 1 = \csc^2 \theta$$



The Third Pythagorean identities is

$$\cot^2 \theta + 1 = \csc^2 \theta$$

Each Pythagorean identity can appear in at least three different forms.

$$\sin^2 \theta + \cos^2 \theta = 1 \Leftrightarrow \sin^2 \theta = 1 - \cos^2 \theta \Leftrightarrow \cos^2 \theta = 1 - \sin^2 \theta$$

$$\sec^2 \theta - \tan^2 \theta = 1 \Leftrightarrow \sec^2 \theta = 1 + \tan^2 \theta \Leftrightarrow \tan^2 \theta = \sec^2 \theta - 1$$

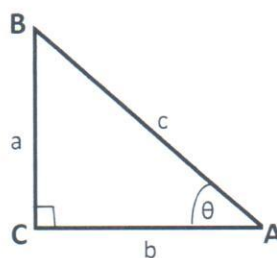
$$\csc^2 \theta - \cot^2 \theta = 1 \Leftrightarrow \csc^2 \theta = 1 + \cot^2 \theta \Leftrightarrow \cot^2 \theta = \csc^2 \theta - 1$$

Now look at the following example problems.

Example 1

Prove $\sin^2 \theta + \cos^2 \theta = 1$.

Solution:



In right $\triangle ABC$, $a^2 + b^2 = c^2$

Divide by c^2 : $\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$

Rename and simplify: $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$

But $\sin \theta = \frac{a}{c}$, $\cos \theta = \frac{b}{c}$ and so on.

Substitute $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$ and get $(\sin \theta)^2 + (\cos \theta)^2 = 1$

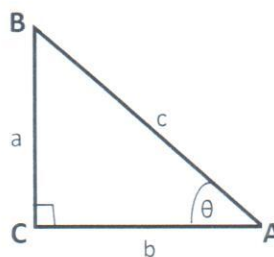
But, $(\sin^2 \theta) = (\sin \theta)^2$ and $(\cos \theta)^2 = (\cos^2 \theta)$

Therefore, $\sin^2 \theta + \cos^2 \theta = 1$.

Example 2

Prove $1 + \tan^2 \theta = \sec^2 \theta$.

Solution:



In right $\triangle ABC$, $a^2 + b^2 = c^2$



Divide by b^2 :

$$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$$

Rename and simplify:

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

But $\tan \theta = \frac{a}{b}$, $\sec \theta = \frac{c}{b}$ and so on.

Substitute $\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$ and get $(\tan \theta)^2 + 1 = (\sec \theta)^2$

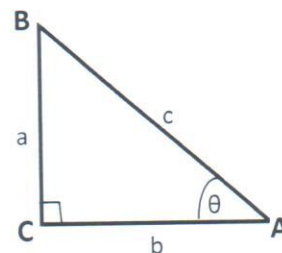
But, $(\tan^2 \theta) = (\tan \theta)^2$ and $(\sec \theta)^2 = (\sec^2 \theta)$

Therefore, $\tan^2 \theta + 1 = \sec^2 \theta$ or $1 + \tan^2 \theta = \sec^2 \theta$.

Example 3

Prove $1 + \cot^2 \theta = \csc^2 \theta$.

Solution:



In right $\triangle ABC$,

$$a^2 + b^2 = c^2$$

Divide by a^2 :

$$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$$

Rename and simplify:

$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

But $\cot \theta = \frac{b}{a}$, $\csc \theta = \frac{c}{a}$ and so on.

Substitute $1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$ and get $1 + (\cot \theta)^2 = (\csc \theta)^2$

But, $(\cot^2 \theta) = (\cot \theta)^2$ and $(\csc \theta)^2 = (\csc^2 \theta)$

Therefore, $\cot^2 \theta + 1 = \csc^2 \theta$ or $1 + \cot^2 \theta = \csc^2 \theta$.



Example 4

If $\sin \theta = \frac{5}{6}$ and $\cos \theta < 0$, find $\cos \theta$.

Solution: Using identity that relates to sine and cosine and does not require knowledge of an angle.

$$\sin^2 \theta + \cos^2 \theta = 1$$

Substitute

$$\left(\frac{5}{6}\right)^2 + \cos^2 \theta = 1$$

Transposition

$$\cos^2 \theta = 1 - \left(\frac{5}{6}\right)^2$$

Square the fraction

$$\cos^2 \theta = 1 - \frac{25}{36}$$

Subtract

$$\cos^2 \theta = \frac{36 - 25}{36} = \frac{11}{36}$$

Find the square root of each side

$$\cos \theta = \frac{\sqrt{11}}{6}$$

But $\cos \theta < 0$, so

$$\cos \theta = -\frac{\sqrt{11}}{6}$$

Example 5

Find the values of the other functions of θ if $\tan \theta = -\frac{3}{4}$ and θ terminates in the second quadrant.

Solution: First, use the Second Pythagorean Identity $\sec^2 \theta = 1 + \tan^2 \theta$

Substituting $\tan \theta = -\frac{3}{4}$, we have

$$\sec^2 \theta = 1 + \left(-\frac{3}{4}\right)^2$$

$$\sec^2 \theta = 1 + \frac{9}{16}$$

$$\sec^2 \theta = \frac{25}{16}$$

$$\sec \theta = \frac{5}{4}$$

Since $\sec \theta$ is negative in the 2nd Quadrant, $\sec \theta = -\frac{5}{4}$



Since the reciprocal of secant is cosine, we use $\cos \theta = \frac{1}{\sec \theta}$ to find $\cos \theta$.

$$\cos \theta = \frac{1}{-\frac{5}{4}}$$

$$\cos \theta = -\frac{4}{5}$$

To find $\sin \theta$, use the ratio identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$, Thus we have

$$\sin \theta = \cos \theta \cdot \tan \theta$$

$$\sin \theta = \left(-\frac{4}{5}\right)\left(-\frac{3}{4}\right)$$

$$\sin \theta = \frac{3}{5}$$

Since the reciprocal of sine is cosecant, we use $\csc \theta = \frac{1}{\sin \theta}$, to find $\csc \theta$.

$$\csc \theta = \frac{1}{\frac{3}{5}}$$

$$\csc \theta = \frac{5}{3}$$

Since the reciprocal of tangent is cotangent, we use $\cot \theta = \frac{1}{\tan \theta}$ to find $\cot \theta$.

$$\cot \theta = \frac{1}{-\frac{3}{4}}$$

$$\cot \theta = -\frac{4}{3}$$

Now do the learning activity.

**LEARNING ACTIVITY 12.2.2.1**

20 minutes

Solve the following problem.

1. If $\sin \theta = \frac{5}{6}$ and $\cos \theta < 0$, find $\tan \theta$ using the ratio identity.

2. Express $\frac{\tan \theta + \sec \theta}{1 + \sin \theta}$ in simplest form.

3. What is the value of $7 \sin \theta + 7 \cos \theta$?

4. Find the values of all the other functions of θ , if $\sin \theta = \frac{3}{5}$.

**12.2 2.2: Proving Identities: Reduction to Simplified Form**

In an identity there are two sides of the equation, one side is known as 'left hand side' (LHS) and the other side is known as 'right hand side' (RHS) and to prove the identity we need to use logical steps showing that one side of the equation ends up with the other side of the equation. To do this, start with either the left-hand side (LHS) or the right-hand side (RHS). Choose the side that looks more readily able to be simplified. Re-express the terms using identities so that it takes the form of the other side.

Example 1

Prove that $\frac{\cot^2 \theta - 1}{\cot^2 \theta + 1} = 1 - 2 \sin^2 \theta$

Solution: Starting with LHS

$$\begin{aligned} \frac{\cot^2 \theta - 1}{\cot^2 \theta + 1} &= \frac{\cot^2 \theta - 1}{\csc^2 \theta} && [\text{substituting } 1 + \cot^2 \theta = \csc^2 \theta] \\ &= \frac{\cot^2 \theta + 1 - 2}{\csc^2 \theta} && [\text{re-expressing to form identity}] \\ &= \frac{\csc^2 \theta - 2}{\csc^2 \theta} && [\text{substituting } \csc^2 \theta \text{ for } 1 + \cot^2 \theta] \\ &= \frac{\csc^2 \theta - 2}{\csc^2 \theta} - \frac{2}{\csc^2 \theta} && [\text{separating into two fractions}] \\ &= 1 - 2 \sin^2 \theta && [\text{since } \frac{1}{\csc^2 \theta} = \sin^2 \theta] \\ &= \text{LHS} \end{aligned}$$

Example 2

Prove that $\frac{1 - \sin \theta}{1 + \sin \theta} = (\sec \theta - \tan \theta)^2$

Solution:

Starting with RHS

$$\begin{aligned} (\sec \theta - \tan \theta)^2 &= \left(\frac{1}{\cos \theta} - \frac{\sin \theta}{\cos \theta} \right)^2 && [\text{using definitions of } \sec \theta, \tan \theta] \\ &= \frac{(1 - \sin \theta)^2}{\cos^2 \theta} && [\text{squaring numerator and denominator}] \\ &= \frac{(1 - \sin \theta)^2}{1 - \sin^2 \theta} && [\text{since } \sin^2 \theta + \cos^2 \theta = 1] \Leftrightarrow \cos^2 \theta = 1 - \sin^2 \theta \end{aligned}$$



$$= \frac{(1 - \sin \theta)(1 - \sin \theta)}{(1 - \sin \theta)(1 + \sin \theta)} \quad [\text{factorising and cancelling}]$$

$$= \frac{1 - \sin \theta}{1 + \sin \theta} \quad \text{Proved}$$

Example 3

Prove that $\tan^4 \theta + \tan^2 \theta = \sec^4 \theta - \sec^2 \theta$

Solution:

Starting with the LHS

$$\begin{aligned} \tan^4 \theta + \tan^2 \theta &= \tan^2 \theta (\tan^2 \theta + 1) && [\text{By factoring}] \\ &= (\sec^2 \theta - 1) (\tan^2 \theta + 1) && [\text{since } \tan^2 \theta = \sec^2 \theta - 1] \\ &= (\sec^2 \theta - 1) (\sec^2 \theta) && [\text{since } \tan^2 \theta + 1 = \sec^2 \theta] \\ &= \sec^4 \theta - \sec^2 \theta && [\text{expand}] \\ &= \text{RHS} && \text{Proved} \end{aligned}$$

**LEARNING ACTIVITY 12.2.2.2**

20 minutes

1. Prove each of the following identities by transforming the left side only.

a) $\sin \theta \cot \theta = \cos \theta$

b) $\sec \theta \cot \theta = \csc \theta$



$$\text{c) } (1 + \tan^2 \theta) \sin^2 \theta = \tan^2 \theta$$

$$\text{d) } \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \sec^2 \theta$$

$$\text{e) } \frac{\csc^2 \theta - \cot^2 \theta}{\sec^2 \theta} = \cos^2 \theta$$

12.2.2.3: Addition and Subtraction Identities

In the previous sections, the fundamental identities discussed all involved working with a single variable. In this section, we will discuss identities that use two variables and are known as the **addition and subtraction identities**, **sum and difference identities**, or simply **trigonometric addition identities**.

The sum identity for sine:	$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
----------------------------	--

The difference identity for sine:	$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
-----------------------------------	--

The sum identity for cosine:	$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
------------------------------	--

The difference identity for cosine:	$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
-------------------------------------	--

The sum identity for tangent :	$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
--------------------------------	---

The difference identity for tangent:	$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$
--------------------------------------	---



Now look at the following worked examples.

Example 1

Find the exact value of $\cos 75^\circ$.

Solution:

Express $\cos 75^\circ$ as a sum of two angles: $\cos 75^\circ = \cos (45^\circ + 30^\circ)$

Use the sum formula for cosine: $= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$

Evaluate: $= \frac{\sqrt{2}}{2} \left(\frac{\sqrt{3}}{2} \right) - \frac{\sqrt{2}}{2} \left(\frac{1}{2} \right)$

Simplify; $= \frac{\sqrt{6} - \sqrt{2}}{4}$

Therefore, $\cos 75^\circ = \frac{\sqrt{6} - \sqrt{2}}{4}$.

Example 2

Find the exact value of $\tan \frac{\pi}{12}$.

Solution:

Express $\tan \frac{\pi}{12}$ as a difference of two angles: $\tan \frac{\pi}{12} = \tan \left(\frac{\pi}{3} - \frac{\pi}{4} \right)$

Use the difference formula for tangent: $= \frac{\tan \frac{\pi}{3} - \tan \frac{\pi}{4}}{1 + \tan \frac{\pi}{3} \tan \frac{\pi}{4}}$

Evaluate; $= \frac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)}$

Simplify: $= 2 - \sqrt{3}$

Therefore, $\tan \frac{\pi}{12} = 2 - \sqrt{3}$.



Example 3

Change $\sin 70^\circ \cos 120^\circ + \sin 70^\circ \cos 120^\circ$ into a trigonometric function in single variable and evaluate it.

Solution:

$\sin 70^\circ \cos 120^\circ + \sin 70^\circ \cos 120^\circ$ is one side of the sum identity for sine.

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Complete the sum identity for sine: $\sin 70^\circ \cos 120^\circ + \sin 70^\circ \cos 120^\circ = \sin(70^\circ + 120^\circ)$

Add inside the parentheses: $= \sin(190^\circ)$

190° is a third quadrant angle, so sin is negative: $= -\sin 190^\circ$

To get the reference angle in the 3rd quadrant, subtract 180° : $= -\sin 190^\circ - 180^\circ$

Subtract: $= -\sin 10^\circ$

Evaluate: $= -0.174$

Therefore, $\sin 70^\circ \cos 120^\circ + \sin 70^\circ \cos 120^\circ = -0.174$.

Example 4

Use the sum or difference identity for sine to find the value of $\sin(165^\circ)$.

Solution:

$\sin(165^\circ)$ is a 2nd quadrant angle. ($90^\circ < 165^\circ < 180^\circ$)

$\sin(165^\circ) = \sin(120^\circ + 45^\circ)$ this is one side of the sum identity for sine

Using the sum identity for sine: $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

Complete the sum identity: $\sin(120^\circ + 45^\circ) = \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ$

Evaluate using the table of values on page 30: $= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

Simplify and add. $= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

Answer: $= \frac{\sqrt{6} - \sqrt{2}}{4}$

Therefore, $\sin(165^\circ) = \frac{\sqrt{6} - \sqrt{2}}{4}$.



Example 5

Change $\cos 130^\circ \cos 70^\circ + \sin 130^\circ \sin 70^\circ$ into a trigonometric function in single variable and evaluate it.

Solution:

$\cos 130^\circ \cos 70^\circ + \sin 130^\circ \sin 70^\circ$ is one side of the difference identity for cosine.

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

Complete the difference identity for cosine: $\cos 130^\circ \cos 70^\circ + \sin 130^\circ \sin 70^\circ = \cos(130^\circ - 70^\circ)$

Subtract inside the parentheses: $= \cos(60^\circ)$

All functions are positive in the first quadrant, so evaluate: $\cos 60^\circ = 0.5$ or $\frac{1}{2}$

Example 6

Given that $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$, change $\cos 130^\circ \cos 70^\circ - \sin 130^\circ \sin 70^\circ$ into a trigonometric function in single variable and evaluate it.

Solution:

$\cos 130^\circ \cos 70^\circ - \sin 130^\circ \sin 70^\circ$ is one side of the sum identity for cosine.

Complete the sum identity for cosine: $\cos 130^\circ \cos 70^\circ - \sin 130^\circ \sin 70^\circ = \cos(130^\circ + 70^\circ)$

Add inside the parentheses: $= \cos(200^\circ)$

200° is a third quadrant angle, so cos is negative: $= -\cos 200^\circ$

To get the reference angle in the 3rd quadrant, subtract 180° : $= -\cos 200^\circ - 180^\circ$

Subtract: $= -\cos 20^\circ$

Evaluate: $= -0.94$

Example 7

Given that $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$, find $\cos\left(\frac{5\pi}{12}\right)$.

Solution:

Since $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$,



Then, $\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$ this is one side of the sum identity for cosine.

Complete the sum identity for cosine: $\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

Evaluate using the table of values on page 30: $= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

Multiply: $= \left(\frac{\sqrt{6}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)$

Answer: $= \frac{\sqrt{6} - \sqrt{2}}{4}$

Example 6

If $\tan \alpha = 3$ and $\tan \beta = \frac{1}{2}$ what is the tangent of their sum?

Solution: Use the sum identity for tangent: $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$

Substitute $\tan \alpha = 3$ and $\tan \beta = \frac{1}{2}$: $\tan(\alpha + \beta) = \frac{3 + \frac{1}{2}}{1 - (3)\left(\frac{1}{2}\right)}$

Combine terms: $= \frac{\frac{6+1}{2}}{1 - \frac{3}{2}}$

Simplify: $= \frac{\frac{7}{2}}{-\frac{1}{2}}$

Divide: $= \frac{7}{2} \div -\frac{1}{2} = \frac{7}{2}(-2) = -7$

Answer: $= -7$

Example 7

If $\tan \alpha = 2$ and $\tan \beta = \frac{1}{2}$ what is the tangent of their difference?

Solution: Use the difference identity for tangent: $\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$



Substitute $\tan \alpha = 2$ and $\tan \beta = \frac{1}{2}$:

$$\tan(\alpha - \beta) = \frac{2 - \frac{1}{2}}{1 + (2)\left(\frac{1}{2}\right)}$$

Combine terms:

$$= \frac{4 - 1}{1 + \frac{4}{4}}$$

Simplify:

$$= \frac{3}{2}$$

Divide

$$= \frac{3}{2} \div 2 = \frac{3}{2} \left(\frac{1}{2}\right) = \frac{3}{4}$$

Answer:

$$= \frac{3}{4}$$

**LEARNING ACTIVITY 12.2.2.3**

20 minutes

Solve the following trigonometric functions using the sum and difference identity for sine and cosine.

1. Solve $\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$ to a trigonometric function having a single angle and give its value.
2. Write $\sin\left(x - \frac{\pi}{3}\right)$ in terms of $\sin x$ and $\cos x$.



3. Find the exact value of $\cos 42^\circ \cos 18^\circ - \sin 42^\circ \sin 18^\circ$.

4. Simplify $\sin\left(\frac{\pi}{4} + 2x\right) + \sin\left(\frac{\pi}{4} - 2x\right)$ to a trigonometric function having a single angle

5. If $\tan \alpha = 4$ and $\tan \beta = \frac{1}{3}$, what is the tangent of their sum?

6. If $\tan \alpha = 2$ and $\tan \beta = \frac{1}{4}$, what is the tangent of their difference?

**12.2.2.4: Double Angle Formulas: Functions of $2x$**

The double angle identities for sine and cosine as well as the half angle identities are derived from the sum and difference identities given previously.

Here are the identities:

$$\begin{aligned} \text{The double angle identity for sine:} \quad & \sin 2\theta = 2\sin\theta\cos\theta \\ \text{The double angle identity for cosine:} \quad & \cos 2\theta = \cos^2\theta - \sin^2\theta \\ & \cos 2\theta = 2\cos^2\theta - 1 \\ & \cos 2\theta = 1 - 2\sin^2\theta \end{aligned}$$

These identities are used to express the sine, cosine or tangent on an angle that's double the size of the one you know, in terms of the sine and cosine of that known angle. Notice that there are three times as many forms with cosine to the left of the equal sign as there are for sine in that location.

The **double angle identity for tangent** is derived using the **tangent sum identity**.

The formula is written as:

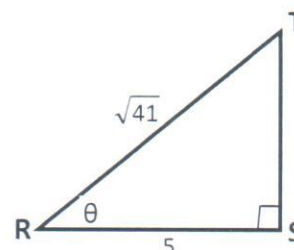
$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

Example 1

In the right $\triangle RST$ at the right, $\cos \theta = \frac{5}{\sqrt{41}}$. Find the value of $\sin \theta$.

Solution:

$$\begin{aligned} \text{Start with the Pythagorean Theorem:} \quad & (TS)^2 = (\sqrt{41})^2 - (5)^2 \\ & (TS)^2 = 41 - 25 \\ & (TS)^2 = 16 \\ & TS = 4 \end{aligned}$$



$$\text{Since } \sin = \frac{\text{opposite}}{\text{hypotenuse}}, \sin \theta = \frac{4}{\sqrt{41}}$$

Now using the double angle identity for sine: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\text{Substitute } \sin \theta = \frac{4}{\sqrt{41}} \text{ and } \cos \theta = \frac{5}{\sqrt{41}}: \quad \sin 2\theta = 2\left(\frac{4}{\sqrt{41}}\right)\left(\frac{5}{\sqrt{41}}\right)$$

$$\text{Group to multiply:} \quad \sin 2\theta = \frac{2(4)(5)}{(\sqrt{41})(\sqrt{41})}$$

$$\text{Answer:} \quad \sin 2\theta = \frac{40}{41}$$



Example 2

If θ is in the second quadrant and $\sin \theta = \frac{3}{5}$, find $\sin 2\theta$.

Solution:

Since $\sin = \frac{\text{opposite}}{\text{hypotenuse}}$ and $\sin \theta = \frac{3}{5}$, we find the adjacent side by using Pythagorean

Theorem.

$$b^2 = c^2 - a^2$$

$$b^2 = 5^2 - 3^2$$

$$b^2 = 25 - 9$$

$$b^2 = 16$$

$$b = 4$$

Since $\sin \theta = \frac{3}{5}$, then $\cos \theta = \frac{4}{5}$. Using the double angle identity for sine:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Substitute $\sin \theta = \frac{3}{5}$, then $\cos \theta = \frac{4}{5}$: $\sin 2\theta = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$

Group to multiply:

$$\sin 2\theta = \frac{2(3)(4)}{5(5)}$$

Answer:

$$\sin 2\theta = \frac{24}{25}$$

Example 3

If $\cos \theta = \frac{3}{5}$ and $\sin \theta < 0$, find $\sin 2\theta$, $\cos 2\theta$ and $\tan 2\theta$.

Solution:

a. To find $\sin 2\theta$, we must first find the value of $\sin \theta$.

Use the Pythagorean identity: $\sin^2 \theta + \cos^2 \theta = 1$,

Substitute $\cos \theta = \frac{3}{5}$: $\sin^2 \theta + \left(\frac{3}{5}\right)^2 = 1$

Simplify: $\sin^2 \theta + \frac{9}{25} = 1$

Transposition: $\sin^2 \theta = 1 - \frac{9}{25}$

Subtract the RHS $\sin^2 \theta = \frac{16}{25}$

Square root property: $\sin \theta = \pm \frac{4}{5}$

Since $\sin \theta < 0$, we choose the negative square root.



Now use the double angle identity for sine: $\sin 2\theta = 2\sin \theta \cos \theta$

Substitute the $\sin \theta = -\frac{4}{5}$ and $\cos \theta = \frac{3}{5}$ $\sin 2\theta = 2\left(-\frac{4}{5}\right)\left(\frac{3}{5}\right)$

Multiply: $= -\frac{24}{25}$

Therefore, $\sin 2\theta = -\frac{24}{25}$.

- b. To find $\cos 2\theta$, use any of the double angle identity for cosine.

Use the first one: $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$

Substitute the $\sin \theta = -\frac{4}{5}$ and $\cos \theta = \frac{3}{5}$: $\cos 2\theta = \left(\frac{3}{5}\right)^2 - \left(-\frac{4}{5}\right)^2$

Simplify the RHS: $\cos 2\theta = \frac{9}{25} - \frac{16}{25}$

Subtract: $\cos 2\theta = -\frac{7}{25}$

Therefore, $\cos 2\theta = -\frac{7}{25}$.

- c. To find $\tan 2\theta$, first find the value of $\tan \theta$ using the ratio identity for tangent.

Hence, we have: $\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{-\frac{4}{5}}{\frac{3}{5}} = \left(-\frac{4}{5}\right)\left(\frac{5}{3}\right) = -\frac{4}{3}$

Now, use the double angle identity for tangent: $\tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta}$

Substitute $\tan \theta = -\frac{4}{3}$ RHS. $\tan 2\theta = \frac{2\left(-\frac{4}{3}\right)}{1 - \left(-\frac{4}{3}\right)^2}$

Multiply: $\tan 2\theta = \frac{-\frac{8}{3}}{1 - \frac{16}{9}}$



Simplify:

$$\tan 2\theta = \frac{-\frac{8}{3}}{-\frac{9}{7}}$$

Divide:

$$\tan 2\theta = \frac{24}{7}$$

Therefore, $\tan 2\theta = \frac{24}{7}$.

Now do the learning activity.



LEARNING ACTIVITY 12.2.2.4



20 minutes

Solve the following using the double angle identity.

1. If θ is a second quadrant angle and $\sin \theta = \frac{5}{13}$, what is $\cos 2\theta$?

2. If θ is a second quadrant angle and $\sin \theta = \frac{8}{10}$, what is $\sin 2\theta$?

3. If $\theta = \arcsin \frac{3}{5}$, find $\tan 2\theta$.



12.2.2.5: Half Angle Identity Formulas: Functions of $\frac{\theta}{2}$

In this section, we will discuss first the half angle formulas for sine and cosine then followed by the half angle formula for tangent.

Half-Angle Identity Formulas for Sine and Cosine

These identities are used to express the sine and cosine of an angle that is half the size of the angle.

Here are the formulas:

The half angle identity for sine:
$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

The half angle formula for cosine:
$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

Notice that each radical sign is preceded by a $\pm$ (which means plus or minus). That is because the sign of the answer will depend on which quadrant the half angle lies in. If $\theta = 280^\circ$, then $\frac{\theta}{2} = 140^\circ$. Since 140° is a second quadrant angle, $\sin \theta$ would be preceded by a plus (+) sign, and $\cos \theta$ would be preceded by a minus (–) sign.

Example 1

If $\cos \theta = -\frac{5}{6}$ and θ is in third quadrant, find $\cos \frac{\theta}{2}$.

Solution:

Since θ is in the 3rd quadrant, then θ is between 270° and 360° . This means $\frac{\theta}{2}$ is greater than 135° and less than 180° . That is in 2nd quadrant. Hence, cosine is also negative in the 2nd quadrant.

Now let us use the half-angle formula for cosine:
$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

Substitute and adjust the sign:

$$\cos \frac{\theta}{2} = -\sqrt{\frac{1 + \left(-\frac{5}{6}\right)}{2}}$$

Add the terms in the numerator:

$$\cos \frac{\theta}{2} = -\sqrt{\frac{16 - 5}{16}} = -\sqrt{\frac{11}{16}}$$



Divide:

$$\cos \frac{\theta}{2} = -\sqrt{\frac{11}{16} \div \frac{2}{1}} = -\sqrt{\frac{11}{16} \times \frac{1}{2}} = -\sqrt{\frac{11}{32}}$$

Work out what is inside the radical sign:

$$\cos \frac{\theta}{2} = -\frac{\sqrt{11}}{4\sqrt{2}}$$

Rationalize the denominator:

$$\cos \frac{\theta}{2} = -\frac{\sqrt{11}}{4\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{22}}{8}$$

$$\text{Therefore, } \cos \frac{\theta}{2} = -\frac{\sqrt{22}}{8}.$$

Example 2

If θ is a positive acute angle and $\cos \theta = \frac{1}{9}$, find $\sin \frac{\theta}{2}$.

Solution:

Use the half-angle formula for sine:

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

Since θ is a positive acute angle, sine is positive:

$$\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}}$$

Substitute $\cos \theta = \frac{1}{9}$:

$$= \sqrt{\frac{1 - \frac{1}{9}}{2}}$$

Subtract the terms in the numerator:

$$\sin \frac{\theta}{2} = \sqrt{\frac{\frac{9-1}{9}}{2}} = \sqrt{\frac{8}{9}}$$

Divide:

$$\sin \frac{\theta}{2} = \sqrt{\frac{8}{9} \div \frac{1}{1}} = \sqrt{\frac{8}{9} \times \frac{1}{1}} = \sqrt{\frac{8}{9}}$$

Work out what is inside the radical sign:

$$\sin \frac{\theta}{2} = \sqrt{\frac{4}{9}} = \frac{2}{3}$$

$$\text{Therefore, } \sin \frac{\theta}{2} = \frac{2}{3}.$$



Half-angle Identity Formula for Tangent

The half-angle identity formula for tangent has three different forms. In the third form, its sign is determined by the quadrant in which the angle terminates.

Here are the formulas:

$$\tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}$$

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$$

$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

Example 3

If the cosine of a third quadrant angle is $-\frac{2}{3}$, find the tangent of an angle half that size.

Solution:

Select the half angle formula for tangent that applies: $\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$

Since the cosine is negative and it is in the 3rd quadrant, tangent in the 3rd quadrant is positive.

Hence we use the positive half-angle formula:

$$\tan \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

Substitute $\cos = -\frac{2}{3}$:

$$\tan \frac{\theta}{2} = \sqrt{\frac{1 - \left(-\frac{2}{3}\right)}{1 + \left(-\frac{2}{3}\right)}}$$

Simplify :

$$\tan \frac{\theta}{2} = \sqrt{\frac{\frac{5}{3}}{\frac{1}{3}}}$$

Divide:

$$\tan \frac{\theta}{2} = \sqrt{\frac{5}{3} \div \frac{1}{3}} = \sqrt{\frac{5}{3} \times \frac{3}{1}} = \sqrt{5}$$

Therefore, $\tan \frac{\theta}{2} = \sqrt{5}$

**LEARNING ACTIVITY 12.2.2.5**

20 minutes

Solve the following using half-angle formula.

1. If $\cos \theta = -\frac{3}{8}$, and θ is in the third quadrant, find $\cos \frac{1}{2}\theta$.
2. Find the exact value of $\tan \frac{\pi}{8}$ using the half-angle formula.
3. If the cosine of a third quadrant angle is $-\frac{3}{4}$, find the tangent of an angle half that size.
4. If θ is a positive acute angle and $\cos \theta = \frac{2}{9}$, find $\sin \frac{\theta}{2}$.



12.2.2.6: Trigonometric Equations

In this section we will look at the solution of trigonometric equations using trigonometric identities.

Again let us define a trigonometric equation.

An equation that contains trigonometric functions is called a **trigonometric equation**.

Examples:

- a. $\sin^2 x + \cos^2 x = 1$
- b. $2 \sin x - 1 = 0$
- c. $\tan^2 2x - 1 = 0$

Solving Trigonometric Equations

To solve a trigonometric equation, we use the rules of algebra to isolate the trigonometric function on one side of the equal sign. Then we use our knowledge of the values of the trigonometric functions to solve for the variable.

When you solve a trigonometric equation that involves only one trigonometric expression, begin by isolating the expression.

When trigonometric functions cannot be combined on one side of an equation, try to factor the equation and then apply **zero product property** to solve the equation. If the equation has quadratic form, first factor if possible. If not possible, apply the **quadratic formula** to solve the equation.

Solution of Trigonometric Equations

If a given equation has one solution, it has in general an unlimited number of solutions.

Example 1

Find the solution of $\cos \theta = 0$.

Solution:

The complete solution of $\cos \theta = 0$ is given by $\theta = \frac{\pi}{2} + 2n\pi$, $\theta = \frac{3\pi}{2} + 2n\pi$, where n is an integer.

Note: We shall list only the particular solution for which $0 \leq \theta < 2\pi$.



There is no general method for solving trigonometric equations. However, three standard procedures are illustrated below and other procedures are introduced in the solution of some problems.

1. Equations may be factorable.
2. Equations may be expressed in terms of a single function.
3. Both sides of the equation are squared.

Example 1

Solve $\sin^2 \theta + \sin \theta - 2 = 0$

Solution: This may be factorable

Write the original equation:

$$\sin^2 \theta + \sin \theta - 2 = 0,$$

Factoring we have:

$$(\sin \theta + 2)(\sin \theta - 1) = 0$$

Zero-product property:

$$\sin \theta + 2 = 0 ; \sin \theta - 1 = 0$$

Then,

$$\sin \theta + 2 = 0$$

$\sin \theta = -2$ and there is no solution.

From

$$\sin \theta - 1 = 0$$

$$\sin \theta = 1 \text{ and } \theta = \frac{\pi}{2}$$

Example 2

Solve $2 \tan^2 \theta + \sec^2 \theta = 2$ in an interval $0 \leq \theta \leq 2\pi$.

Solution: This equation may be expressed in terms of a single function.

Rewrite the original equation:

$$2 \tan^2 \theta + \sec^2 \theta = 2$$

Replace $\sec^2 \theta$ by $1 + \tan^2 \theta$:

$$2 \tan^2 \theta + 1 + \tan^2 \theta = 2$$

Then, combine like terms:

$$3 \tan^2 \theta = 1$$

Divide both sides by 3:

$$\tan^2 \theta = \frac{1}{3}$$

Find the square root of both sides:

$$\tan \theta = \pm \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

Rationalise the denominator:

$$\tan \theta = \pm \frac{\sqrt{3}}{3}$$

From $\tan \theta = \frac{\sqrt{3}}{3}$, $\theta = \frac{\pi}{6}$ and $\frac{7\pi}{6}$

From $\tan \theta = -\frac{\sqrt{3}}{3}$, $\theta = \frac{5\pi}{6}$ and $\frac{11\pi}{6}$

The required solutions are $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

**Example 3**

Solve $\sin \theta + \cos \theta = 1$ in an interval $0 \leq \theta \leq 2\pi$.

Solution: This equation can be solved by squaring both sides.

By transposition, $\sin \theta + \cos \theta = 1$,

we have $\sin \theta = 1 - \cos \theta$

Squaring both sides, we get $\sin^2 \theta = 1 - 2 \cos \theta + \cos^2 \theta$

Then, $1 - \cos^2 \theta = 1 - 2 \cos \theta + \cos^2 \theta$

Thus, $2 \cos^2 \theta - 2 \cos \theta = 0$

Factor the left side $2 \cos \theta (\cos \theta - 1) = 0$

Equating to 0 $2 \cos \theta = 0$; $\cos \theta - 1 = 0$

$$\cos \theta = 0; \cos \theta = 1$$

Therefore, from $\cos \theta = 0$, $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

And from $\cos \theta = 1$, $\theta = 0$.

Example 4

Solve $\tan \theta + \cot \theta = 2 \sec \theta$ in an interval $0 \leq \theta \leq 2\pi$.

Solution: $\tan \theta + \cot \theta = 2 \sec \theta$

By identities, $\tan \theta = \frac{\sin \theta}{\cos \theta}$, $\cot \theta = \frac{\cos \theta}{\sin \theta}$ and $\sec \theta = \frac{1}{\cos \theta}$

Substitute these in the equation: $\frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} = 2 \left(\frac{1}{\cos \theta} \right)$

Multiply by $\cos \theta \sin \theta$: $\sin^2 \theta + \cos^2 \theta = 2 \sin \theta$

By Pythagorean identities: $1 = 2 \sin \theta$

Then, $\sin \theta = \frac{1}{2}$. Therefore, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$



Example 5

Solve $2 \sin \theta - 1 = 0$ in an interval $0 \leq \theta \leq 2\pi$.

Solution: First isolate $\sin \theta$ on one side of the equation.

Rewrite the original equation: $2 \sin \theta - 1 = 0$

Add 1 to each side: $2 \sin \theta = 1$

Divide each side by 2: $\sin \theta = \frac{1}{2}$

Therefore, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Example 6

Solve $4 \tan^2 \theta - 1 = 0$

Solution: First isolate $\sin \theta$ on one side of the equation.

Rewrite the original equation: $4 \tan^2 \theta - 1 = 0$

Add 1 to each side: $4 \tan^2 \theta = 1$

Divide each side by 4: $\tan^2 \theta = \frac{1}{4}$

Take the square root of each side: $\tan \theta = \pm \frac{1}{2}$ or ± 0.5

Use the calculator to find θ for which $\tan \theta = \pm \frac{1}{2}$ or ± 0.5

Therefore, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

Now do the learning activity on the next page.

**LEARNING ACTIVITY 12.2.2.6**

20 minutes

Solve the following trigonometric equations.

1. If $0 \leq \theta \leq 2\pi$, solve the equation:

a. $4 \tan \theta + 2 = 2 \tan \theta.$

b. $2(\cos x + 1) = 1$

c. $2 \cos x + 3 = 0$

d. $4 \sin^2 x - 1 = 0$



2. If $0 \leq x \leq \pi$, solve $\tan^2 x - 3 = 0$.

3. Solve $\tan^2 x + \sec x - 1 = 0$, when $0 \leq x \leq 2\pi$.

SUMMATIVE TASK 12.2.2

Workout the following exercises by applying what you have studied about fundamental identities.

1. Find the values of all the function of θ from the given condition if $\cos \theta = -\frac{4}{5}$.

2. By means of fundamental identities, express the following in terms of $\sin \theta$ and $\cos \theta$.

a. $\cot \theta + \csc \theta$

b. $\sin \theta \cot \theta + \csc \theta$



By means of fundamental identities, express the following in terms of $\sin \theta$ only.

a. $2 \cos^2 \theta - 1$

b. $\csc \theta \cot \theta - \sec \theta$

c. $\sec^2 \theta + \tan^2 \theta$

d. $\sec \theta \tan \theta + \csc \theta$

4. By means of fundamental identities, prove each of the following identities by transforming the left side only.

a. $\cos \theta \tan \theta = \sin \theta$

b. $\sin \theta \sec \theta = \tan \theta$

c. $\cot \theta \cos \theta = \csc \theta - \sin \theta$

d. $\frac{\tan \theta - 1}{\tan \theta + 1} = \frac{1 - \cot \theta}{1 + \cot \theta}$



5. Using addition and subtraction identities, find the exact values of the following:

a. $\sin(60^\circ + 30^\circ)$

b. $\sin(45^\circ + 45^\circ)$

6. If $\sin \theta = \frac{3}{4}$, find $\cos 2\theta$ in exact form.

7. Use identities to find the exact value of each of the following

a. $\sin \frac{\pi}{12}$

b. $\tan\left(-\frac{7\pi}{12}\right)$

c. $\frac{\tan 80^\circ + \tan 55^\circ}{1 - \tan 80^\circ \tan 55^\circ}$

d. $\frac{\tan 80^\circ - \tan(-55^\circ)}{1 + \tan 80^\circ \tan(-55^\circ)}$



12.2. 3: VECTORS

Vectors are an absolutely essential 'tool' in physics and a very important part of mathematics. In this section, vectors and their basic components will be defined and quantified. For this section we will concentrate on 2-dimensional vectors.

12.2.3.1: Introduction to Vectors

In real life, many measures such as length, mass, age, time and volume have magnitude (a fancy word for size) alone. Such measures are called **scalar quantities**.

A quantity with magnitude alone, but no direction, is called a **scalar quantity**.

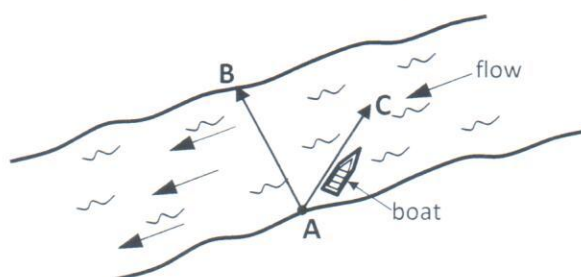
One example of a scalar is distance. This tells us how far we are from a fixed point, but does not give us any information about the direction. Another example of a scalar quantity is the mass of an object.

Other measures, such as force and velocity have both magnitude and direction are called **vectors**, and are typically represented by arrows.

A vector is a quantity that has **size** and **direction**. The fancy word for size is "magnitude". Thus we say, **a vector is a quantity that has both magnitude and direction**.

One common use of vectors is finding the actual direction and speed at which one would have row a boat in order to get across a river to a certain point when the river is flowing downstream at a certain velocity.

For example, in the figure below, to get from point **A** to point **B**, you would row along $\overrightarrow{AC}$.



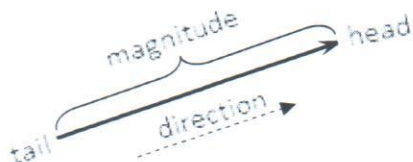
To get from **A** to **B**, row along $\overrightarrow{AC}$.

Notice that the vector $\overrightarrow{AC}$ is written the same way as a ray would be represented in geometry. You need to know whether you are dealing with rays or vectors. In vector notation, the directed line segment $\overrightarrow{AC}$ has an initial point **A** and a terminal point **C**.



Although there are no doubts to seeing speed limits signs on highways, speed is a scalar quantity not a vector quantity. Velocity is a vector since it has both speed and direction.

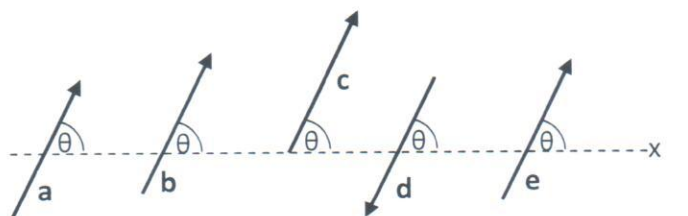
Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector is from its tail to its head.



The **length of the line** shows its **magnitude** and the **arrowhead** points in the **direction**.

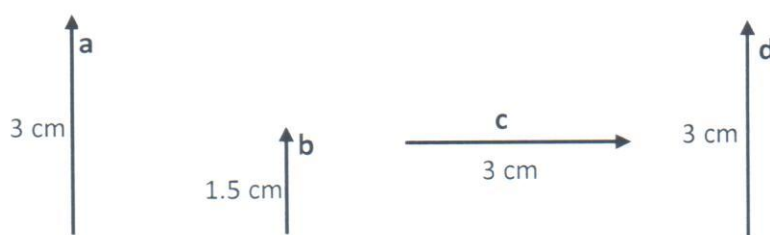
In this section, we will use a boldface letter to represent vectors such as **u** and **v**. Two vectors have the same direction if they are parallel and point the same way.

In the figure below, vectors **a**, **b**, **c** and **e** are **equivalent vectors**. That is they are all of the same magnitude and direction. Vector **d**, however is an **opposite vector** of the same magnitude, parallel to the four others, but pointing in the opposite direction.



Four equivalent vectors and one opposite vector

Example 1



Vector **a** has direction 'up' and a magnitude of 3 cm.

Vector **b** has the same direction as **a**, and has half the magnitude (1.5 cm).

Vector **c** has the same magnitude as **a** (3 cm), but it has different **direction**.

Vector **d** is equivalent to vector **a**. It has the same magnitude and the same direction. It doesn't matter that **a** is in a different position to **d** they are still considered to be **equivalent vectors** because they have the same magnitude and same direction. We can write: **a = d**

Note: We **cannot** write **a = c** because even though **a** and **c** have the same magnitude (4 cm), they have different direction. They are not equivalent.



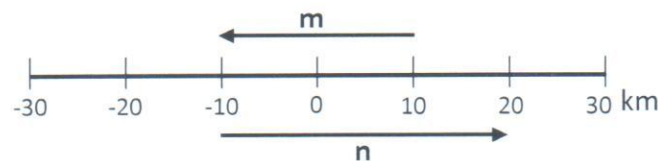
What does it mean if, for two vectors, $\mathbf{a} = \mathbf{c}$? This means first that the length of $\mathbf{a}$ equals the length of $\mathbf{c}$, so that the two vectors have the same magnitude. But it also means that $\mathbf{a}$ and $\mathbf{c}$ are in the same direction. If two vectors are “in the same direction”, then they are **parallel**. We write this down as $\mathbf{a} // \mathbf{b}$.

The following example problems make reference to directions with respect to the points on a compass. Conventionally, the points of the compass show north, as up, south as down, east as right and west as left.

Example 2

Draw a vector $\mathbf{m}$ to represent a wind blowing west at 20 kilometres per hour for one hour, and a vector $\mathbf{n}$ representing a car travelling east that has gone 30 kilometres.

Solution:



Note that since a starting point was not specified for either vector, it may begin anywhere as long as it moves in the specified direction and is of the specified magnitude.

Example 3

Which of the following may be represented by vectors?

- a. the wind
- b. the velocity of a meteor
- c. your height
- d. your weight
- e. the time school begins
- f. your pulse
- g. a line segment on a map connecting Goroka and Lae
- h. a thrown football

Answers: **a, b, d and f**

- Solution:
- a. The wind has magnitude (speed) and direction.
 - b. Velocity has already been mentioned as having speed and direction.
 - c. Your height can be measured whether you are standing or lying down. It has no direction.
 - d. Try putting a scale on the wall and weighing yourself. Weight is a function of gravity with a magnitude and a direction (down).
 - e-g all have magnitude but no direction. The line segment was not pointing to either end.
 - h. A thrown football has magnitude (speed) and direction.

**LEARNING ACTIVITY 12.2.3.1**

20 minutes

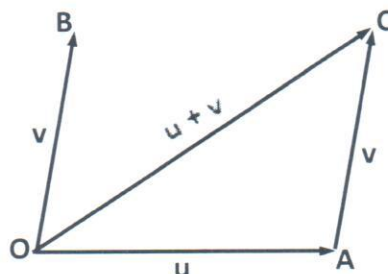
Solve the following problems.

1. Draw a vector **a** to represent a wind blowing north at 35 kilometres per hour for one hour and a vector **b** representing a motorist travelling west who has gone 35 kilometres.
2. Draw a vector **w** representing a weight of 42 grams.
3. What is the difference between scalar versus vector quantities and give examples.



12.3.3.2: Vector Addition Triangle/The Tip-Tail Rule

Vector addition may be shown in two different ways. The following figure shows a triangle of vectors representing two forces acting on an object at O . $\vec{OB}$ represents a force pulling up and slightly to the right, v . $\vec{OA}$ is a force pulling on the same object due right, u .



The tip-tail rule illustrated.

The tip-tail rule says:

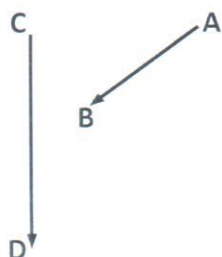
To add two vectors, place the tip of one to the tail of an equivalent of the other, as represented by $\vec{AC}$. Note that $\vec{AC}$ is the same size and goes in the same direction as $\vec{OB}$. It is equivalent to $\vec{OB}$ but has been moved to place its tail at the tip of $\vec{OA}$. The result of this vector addition is $\vec{OC}$, known as the **resultant**, which connects the initial point of $\vec{OA}$ to the terminal point of $\vec{AC}$. $\vec{OA}$ and $\vec{OB}$ are referred to as the **components**.

To better understand vector addition, consider a city block with a single large building on it. You want to travel from O to C , but the building is in the way, so you must go around it by way of A . Your trip from O to C via A resulted in your ending up where you would have been if the building had not been in the way.

Also remember the rule that says two sides of a triangle must add up to a sum greater than the third side is still in effect. When adding two vectors, you are not adding the lengths of the two sides, but the directional sides. The directions play a part in the vector addition and in the direction of the resultant.

Example 1

Draw the resultant of vectors $\vec{AB}$ and $\vec{CD}$ as shown in the following figure.



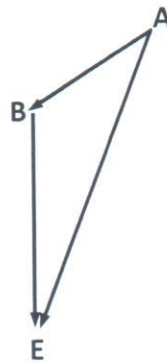


Solution: The solution for Example 1 can be done in two ways.

1. First, draw $\overrightarrow{CD}$. Then, at the tip of $\overrightarrow{CD}$ place $\overrightarrow{DE}$, the equivalent of $\overrightarrow{AB}$. Finally, connect the initial point of $\overrightarrow{CD}$ to the terminal point of $\overrightarrow{DE}$ to get $\overrightarrow{CE}$.

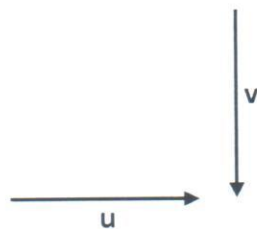


2. Or you could also have started with $\overrightarrow{AB}$ and then place $\overrightarrow{BE}$ (the equivalent of vector $\overrightarrow{CD}$), as in the first figure. The resultant would have been $\overrightarrow{AE}$.

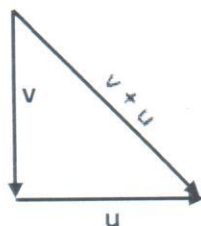


Example 2

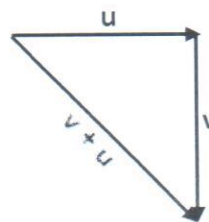
Draw the resultant vectors $\mathbf{v}$ and $\mathbf{u}$ as shown a below.



There are two solutions.



Solution 1



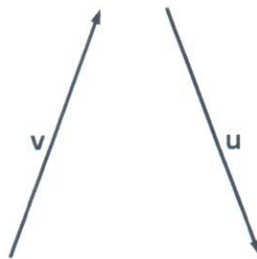
Solution 2



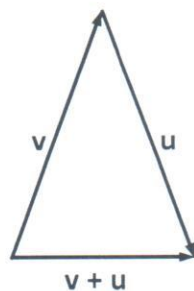
Bring the tip of vector $\mathbf{v}$ to the tail of vector $\mathbf{u}$ or the other way around. In either case, connect the initial point of the first vector to the terminal point of the second to draw the resultant.

Example 3

Draw the resultant of vectors $\mathbf{v}$ and $\mathbf{u}$ as shown in the following figure.



There are two solutions.



Solution 1



Solution 2



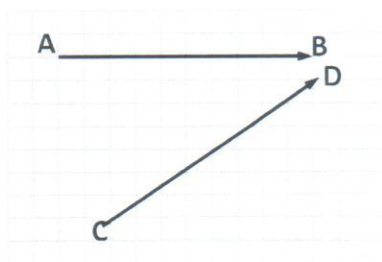
LEARNING ACTIVITY 12.2.3.2



20 minutes

1. Draw the resultant ($\mathbf{r}$) of vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ as shown in the following figure.

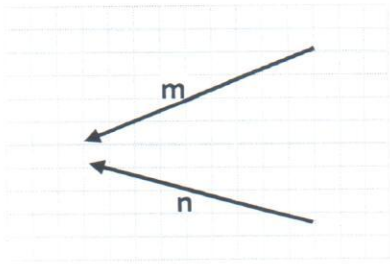
Answer:





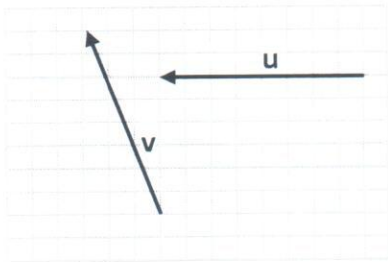
2. Draw the resultant ($\mathbf{r}$) of vectors $\mathbf{m}$ and $\mathbf{n}$ as shown in the following figure.

Answer:



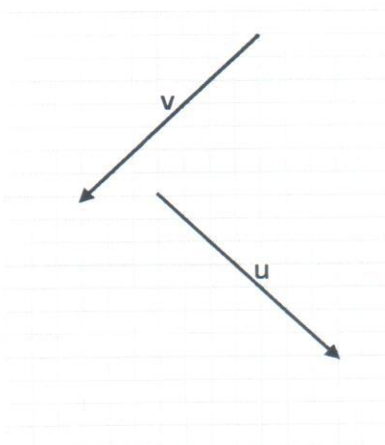
3. Draw the resultant ($\mathbf{r}$) of vectors $\mathbf{v}$ and $\mathbf{u}$ as shown in the following figure.

Answer:



4. Draw the resultant ($\mathbf{r}$) of vectors as shown in the figure.

Answer:

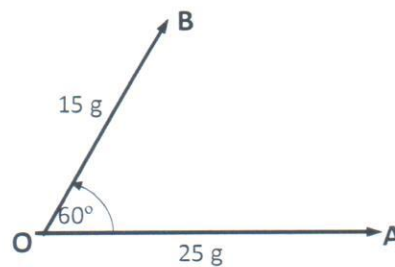




12.3.3.3: Vector Parallelogram: Parallelogram of Forces

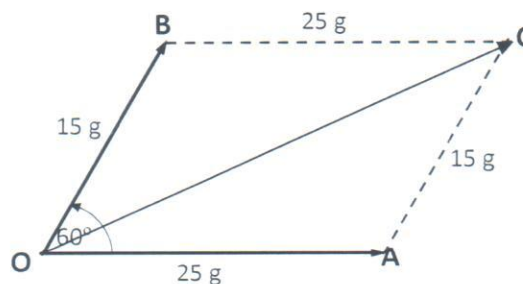
In the previous section, you learned to draw the resultant of two vectors using the tip-tail rule or the vector triangle. In this section, you will learn to illustrate the vector parallelogram or as it is commonly known as the parallelogram of forces.

Consider a situation in which two forces are acting on a object **O**, one component is 25 grams; the other component is 15 grams; and there is a 60° angle between the forces. See the figure below.



Two forces acting at a 60° angle.

By drawing $\overline{AC} \parallel \overline{OB}$ AND $\overline{BC} \parallel \overline{OA}$, a parallelogram will be formed. The diagonal of that parallelogram $\overline{OC}$, is the resultant of the two components, as shown in the following figure.



The diagonal **OC** from the initial points of the vectors is the resultant.

Now recall the Law of Cosines which you studied earlier in section 12.3.1.6.2.

Here is the Law again.

$$a^2 = b^2 + c^2 - 2bc \cos A \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

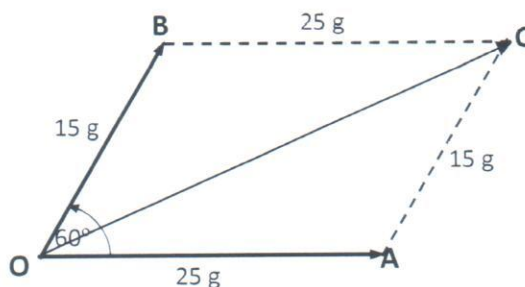
You can use the law to determine the magnitude of the resultant **OC** of the above figure.

See example problems on the next page.



Example 1

Solve the parallelogram forces in the figure.



Solution:

Remember that the Law of Cosines can be used with any general triangle. The 60° angle between the vectors is not usable, since it is not contained wholly within either triangle. However, we can find the reference angle of 60° . Since consecutive angles of a parallelogram are supplementary. Using this fact, we determine the measure of $\angle A$.

$$\begin{aligned}\text{Hence we have,} \quad m\angle O + m\angle A &= 180^\circ \\ 60^\circ + m\angle A &= 180^\circ \\ m\angle A &= 180^\circ - 60^\circ \\ m\angle A &= 120^\circ\end{aligned}$$

This gives us the reference angle of 60° , since $180^\circ - 120^\circ = 60^\circ$, and cosine is negative in the second quadrant.

Using the Cosine Rule:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Substitute $\cos 60^\circ = -\frac{1}{2}$:

$$-\frac{1}{2} = \frac{15^2 + 25^2 - a^2}{2(15)(25)}$$

Multiply:

$$-\frac{1}{2} = \frac{225 + 625 - a^2}{750}$$

Combine like terms:

$$-\frac{1}{2} = \frac{850 - a^2}{750}$$

Cross multiply:

$$2(850 - a^2) = -750$$

Divide both sides by 2:

$$850 - a^2 = -325$$

Transposition:

$$-a^2 = -325 - 850$$

Collect like terms and divide by -1 :

$$a^2 = 1225$$

Square root of both sides:

$$a = 35 \text{ grams}$$

Therefore, the resultant $\overline{OC}$ is 35 grams in magnitude.



But still we need to find the measure of $\angle COA$. To find it, use the Sine Rule..

Here is the Sine Rule. $\frac{a}{\sin A} = \frac{b}{\sin B}$

Substitute $a = 35$, $b = 15$, $A = 120$ and $B = \angle COA$: $\frac{35}{\sin 120^\circ} = \frac{15}{\sin \angle COA}$

Cross multiply: $35 \sin \angle COA = 15 \sin 120^\circ$

The reference angle for 120° in 2^{nd} quadrant is 60° and sine is positive in the 2^{nd} quadrant.

So, $35 \sin \angle COA = 15 \sin 60^\circ$

Divide both sides by 35: $\sin \angle COA = \frac{15 \sin 60^\circ}{35}$

Substitute the value of $\sin 60^\circ = 0.867$ $\sin \angle COA = \frac{15(0.867)}{35}$

Divide by 35: $\sin \angle COA = 0.372$

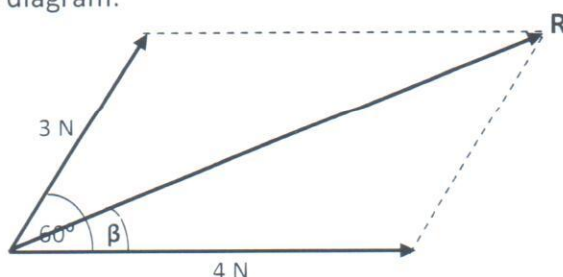
Find $\arcsin 0.372$ or $\sin^{-1} 0.372$ $\angle COA = 21.8^\circ$

Therefore, $\angle COA = 21.8^\circ$.

Example 2

Two forces of 3 N and 4 N are acting at a point such that the angle between them is 60 degrees. Find the resultant force.

Solution: First, draw the diagram.



Using the Cosine Rule : $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Substitute $\cos 60^\circ = \frac{1}{2}$: $-\frac{1}{2} = \frac{3^2 + 4^2 - a^2}{2(3)(4)}$

Multiply: $-\frac{1}{2} = \frac{9 + 16 - a^2}{24}$

Combine like terms: $-\frac{1}{2} = \frac{35 - a^2}{24}$



Cross multiply: $2(35 - a^2) = -24$

Divide both sides by 2: $35 - a^2 = -12$

Transposition: $-a^2 = -12 - 35$

Collect like terms and divide by -1 : $a^2 = 37$

Square root of both sides: $a = 6.08$

So the resultant R is 6.08 N in magnitude.

The direction of R is given by finding the angle β .

Using the Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B}$

Substitute $a = 6.08$, $b = 3$, $A = 120^\circ$ and $B = \beta$: $\frac{6.08}{\sin 120^\circ} = \frac{3}{\sin \beta}$

Cross multiply: $6.08 \sin \beta = 3 \sin 120^\circ$

The reference angle for 120° in 2nd quadrant is 60° and sine is positive in the 2nd quadrant.

So, $6.08 \sin \beta = 3 \sin 60^\circ$

Divide both sides by 6.08: $\sin \beta = \frac{3 \sin 60^\circ}{6.08}$

Substitute the value of $\sin 60^\circ = 0.867$ $\sin \beta = \frac{3(0.867)}{6.08}$

Divide by 6.08: $\sin \beta = \frac{2.601}{6.08}$

$$\sin \beta = 0.428$$

Find $\arcsin 0.428$ or $\sin^{-1} 0.428$ $\beta = 25.3^\circ$

Therefore, $\beta = 22.3^\circ$.

Thus, resultant R is 6.08 N in magnitude and is at an angle of 25.3° to the 4 N force.

Example 3

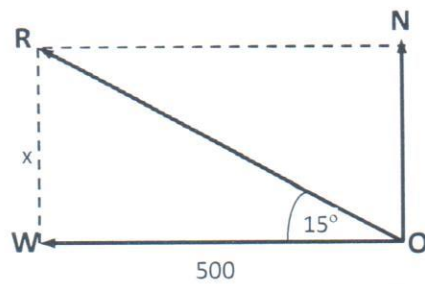
An airplane is trying to fly due west at 500 km per hour. Because of a wind from the south, the plane's actual path is 15° north of west. Assuming the wind to be blowing at a steady pace, what is its magnitude, to the nearest whole kilometre?

Solution:

First, we start by drawing the diagram, as shown in the figure on the next page where $\mathbf{x}$ is the equivalent vector of the wind, $\overrightarrow{ON}$.



Here is the diagram:



The plane is blown north of west.

With respect to the angle at **O**, the relevant relationship or formula to be used is tangent.

Stating the equation: $\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}}$

Substitute the known values: $\tan 15^\circ = \frac{x}{500}$

Cross multiply: $x = 500 \tan 15^\circ$

Find the value of $\tan 15^\circ$: $x = 500(0.268)$ to 3 decimal point

Multiply: $x = 134$ km per hour

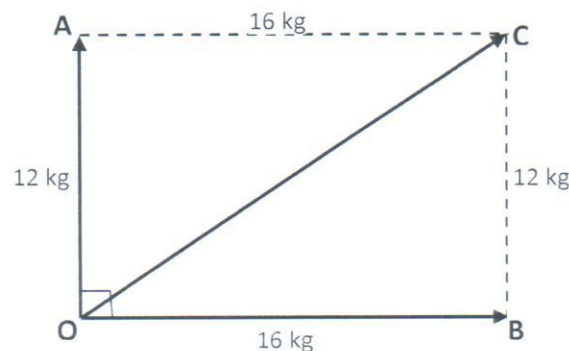
Therefore, the magnitude of the wind is 134 km per hour.

Example 4

A force of 12 kg and a force of 16 kg are acting on an object at an angle of 90 degrees. What are the magnitude and angle of the resultant with respect to the 16 kg force?

Solution:

Draw the diagram:



Forces acting at right angles.



In the figure, $\overrightarrow{OB}$ represents the 16 kg force
 $\overrightarrow{OA}$ represents the 12 kg force
O is the object upon which the forces are acting
 $\overrightarrow{OC}$ is the resultant force

Notice that the corners of this parallelogram are right angles, so the parallelogram of forces is, in fact a rectangle of forces. This means that you can find the resultant force $\overrightarrow{OC}$ using the Pythagorean Theorem.

Stating the rule: $c^2 = a^2 + b^2$

Substitute: $\overrightarrow{OA} = 12$ kg to a $c^2 = 12^2 + 16^2$

and $\overrightarrow{OB} = 16$ kg to b

Square the terms on the right side: $c^2 = 144 + 256$

Add like terms: $c^2 = 400$

Take the square root of both sides: $c = 20$

So, the resultant is 20 kg in magnitude.

To find its angle with respect to the 16 kg force, find $\angle COB$.

You know that the tangent of $\angle COB = \frac{12 \text{ kg}}{16 \text{ kg}}$

Divide: $\angle COB = 0.75$

Find $\arctan 0.75$ or $\tan^{-1} 0.75$: $\angle COB = 36.9^\circ$

Therefore, the resultant is 20 kg in magnitude and is at an angle is 36.9° to the 16 kg force.

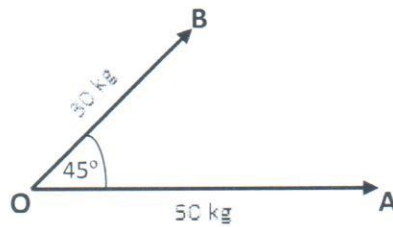
Now do the learning activities on the next page.

**LEARNING ACTIVITY 12.2.3.3**

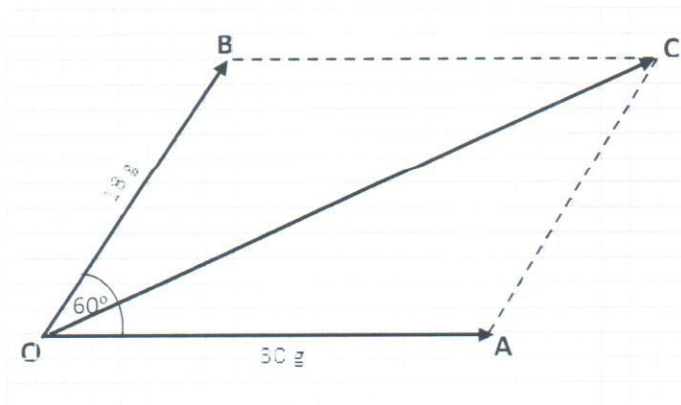
20 minutes

Solve the following problems;

1. Solve the following figure using the parallelogram of forces.



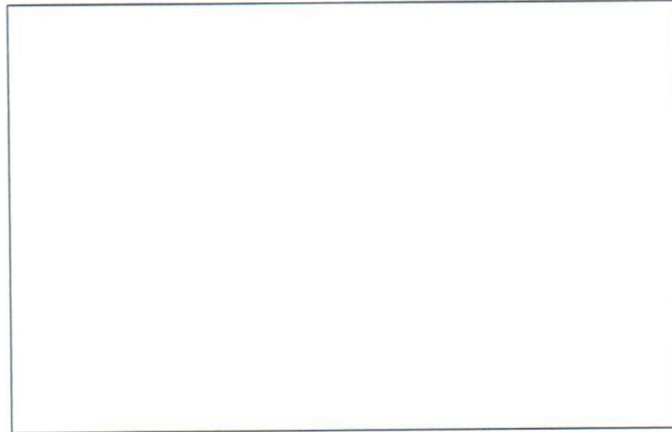
2. Solve the parallelogram of forces in the figure below.





3. A force of 9 kg and a force of 12 kg are acting on an object at an angle of 90° . What are the magnitude and angle of the resultant with respect to the 12 kg force?

Draw the diagram.



Answer: 15 kg at 37°

4. An airplane is trying to fly due east at 400 km per hour. Because of a wind from the south, the plane's actual path is 25° north of east. Assuming the wind to be blowing at a steady pace, what is its magnitude, to the nearest whole km per hour?

Draw the diagram.



Answer: 186.4 kph

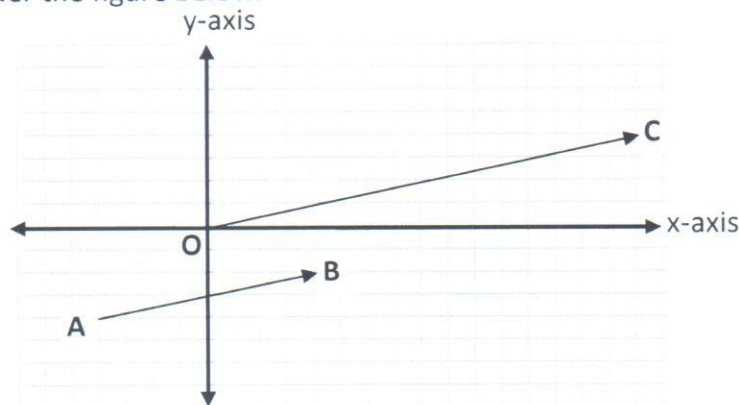


12.2.3.4: Vectors in Rectangular Coordinate System

If a vector is drawn so that its initial point is at the origin of the Cartesian coordinate plane, it is said to be in the standard position.

- A vector in the coordinate plane that has a starting point at the origin is called **standard vector** or sometimes named as **position vector** or **radius vector**.
- Any vector in the coordinate plane that does not have a starting point at the origin is called a **free vector**.

For example, consider the figure below.



In the figure, vector $\overline{AB}$ is equivalent to vector $\overline{OC}$. Since $\overline{OC}$ has its initial point at the origin, it is said to be the **standard vector** for $\overline{AB}$. Vector $\overline{OC}$ is the standard vector for all vectors in the plane that have the same direction and magnitude. In order to identify a standard vector in the coordinate plane, only the coordinates of the terminal point need to be named, since the original point is at the origin. For example, if the coordinates of point **A** are (x_a, y_a) and the coordinates of point **B** are (x_b, y_b) , then the coordinates of $\overline{OC}$ are $(x_b - x_a, y_b - y_a)$.

In case of the figure above, the coordinates of point **A** are $(-5, -4)$, the coordinates of point **B** are $(4, -2)$. Therefore the coordinates of point **C** are $(4 - (-5), -2 - (-4))$, which computes to $(9, 2)$.

Example 1

The endpoints of vector $\overline{AB}$ are at **A** $(-3, -5)$ and **B** $(4, 7)$. What are the coordinates of vector $\overline{OP}$, if $\overline{OP} = \overline{AB}$?

Solution: Remember coordinates of point **A** are (x_a, y_a) and the coordinates of point **B** are (x_b, y_b) , then the coordinates of **P** are $(x_b - x_a, y_b - y_a)$.

Substitute: $[4 - (-3), 7 - (-5)]$

Add: $(4 + 3, 7 + 5)$

Answer: **$(7, 12)$**

That of course, is the terminal point of $\overline{OP}$, its initial point being at the origin.



Example 2

The endpoints of vector $\overline{CD}$ are at **C** (-8, -6) and **D** (-1, 3). What are the coordinates of vector $\overline{OM}$, if $\overline{OM} = \overline{CD}$?

Solution: Remember coordinates of point **C** are (x_a, y_a) and the coordinates of point **D** are (x_b, y_b) , then the coordinates of **M** are $(x_b - x_a, y_b - y_a)$.

Substitute: $[(-1) - (-8), 3 - (-6)]$

Add: $(-1 + 8, 3 + 6)$

Answer: **(7, 9)**

That of course, is the terminal point of $\overline{OM}$, its initial point being at the origin.

Example 3

The endpoints of vector $\overline{AB}$ are at **A** (3, -5) and **B** (11, -9). What are the coordinates of vector $\overline{OP}$, if $\overline{OP} = \overline{AB}$?

Solution: Remember coordinates of point **A** are (x_a, y_a) and the coordinates of point **B** are (x_b, y_b) , then the coordinates of **P** are $(x_b - x_a, y_b - y_a)$.

Substitute: $(11 - 3, -9 - (-5))$

Add: $(8, -9 + 5)$

Answer: **(8, -4)**

That of course, is the terminal point of $\overline{OP}$, its initial point being at the origin.

Example 4

The endpoints of vector $\overline{AB}$ are at **A** (-5, -8) and **B** (4, 7). What are the coordinates of vector $\overline{OP}$, if $\overline{OP} = \overline{AB}$?

Solution: Remember coordinates of point **A** are (x_a, y_a) and the coordinates of point **B** are (x_b, y_b) , then the coordinates of **P** are $(x_b - x_a, y_b - y_a)$.

Substitute: $(4 - (-5), 7 - (-8))$

Add: $(4 + 5, 7 + 8)$

Answer: **(9, 15)**

That of course, is the terminal point of $\overline{OP}$, its initial point being at the origin.

Now do the learning activity on the next page.

**LEARNING ACTIVITY 12.2.3.4**

20 minutes

Solve the following:

1. The endpoints of vector $\overline{AB}$ are at A (-12, -8) and B (-2, 4). What are the coordinates of vector $\overline{OP}$, if $\overline{OP} = \overline{AB}$?
2. The endpoints of vector $\overline{CD}$ are at C (-5, -7) and D (4, 4). What are the coordinates of vector $\overline{OM}$, if $\overline{OM} = \overline{CD}$?
3. The endpoints of vector $\overline{AB}$ are at A (2, -5) and B (12, 7). What are the coordinates of vector $\overline{OP}$, if $\overline{OP} = \overline{AB}$?
4. The endpoints of vector $\overline{CD}$ are at C (-6, -9) and D (2, 4). What are the coordinates of vector $\overline{OM}$, if $\overline{OM} = \overline{CD}$?



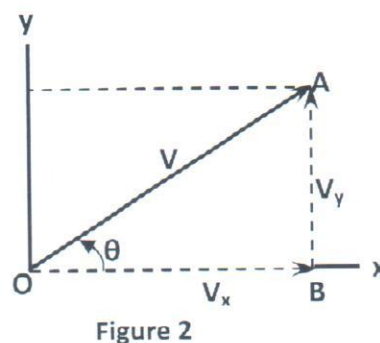
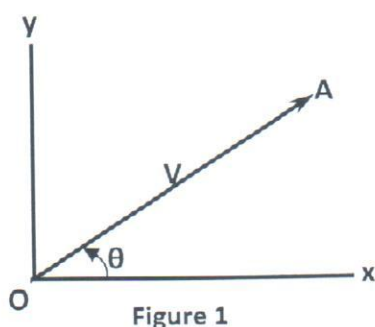
12.2.3.5: Resolution of Vectors

We often think of just a single force acting on an object, but a lifting and a horizontal force can act together to move the object. In this section, we will learn to determine the rectangular components of a given vector.

The process of splitting a vector into various parts or components is called **resolution of vector**. These parts of vectors may act in different directions and are called **components of vector**. We can resolve a vector into a number of components. Here, we will discuss only two components: the X-component and Y-component which are perpendicular to each other. These components are called **rectangular components of vector**.

Methods of resolving a vector into rectangular components

Consider the figures below.



In Figure 1, a vector $\vec{V}$ acting at a point making an angle θ with positive x-axis. Vector $\vec{V}$ is represented by a line OA. From point A draw a perpendicular AB on x-axis as shown in Figure 2. Suppose $\vec{OB}$ and $\vec{BA}$ represents two vectors. Vector $\vec{OB}$ is parallel to the x-axis and vector $\vec{BA}$ is parallel to the y-axis. The magnitudes of these vectors are V_x and V_y respectively. By the method of tip-tail rule, we notice that the sum of these vectors is equal to vector $\vec{V}$. Thus V_x and V_y are the rectangular components of vector $\vec{V}$.

V_x = Horizontal component of $\vec{V}$.

V_y = Vertical component of $\vec{V}$.

Magnitude of Horizontal component V_x : $\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$

$$\cos \theta = \frac{\vec{OB}}{\vec{OA}} = \frac{V_x}{V}$$

$$V_x = V \cos \theta$$



Magnitude of vertical component V_y : $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$

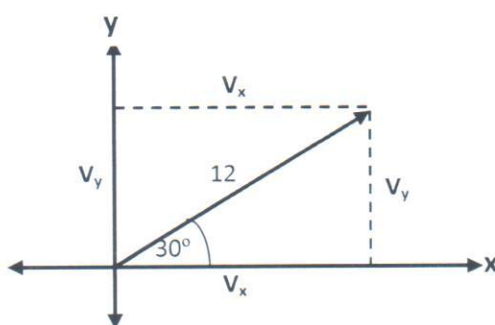
$$\sin \theta = \frac{\overline{AB}}{\overline{OA}} = \frac{V_y}{V}$$

$$V_y = V \sin \theta$$

Example 1

Find the rectangular components of a vector with magnitude of 12 acting at a point making a 30 degrees angle with the x- axis to the nearest integer.

Solution:



Start with the vertical component: Use $V_y = V \sin \theta$

Substitute: $V_y = 12 \sin 30^\circ$

Since $\sin 30^\circ = \frac{1}{2}$, we have: $V_y = 12 \left(\frac{1}{2} \right)$

Multiply and divide: $V_y = 6$

Hence, the vertical component vector is 6.

Next, find the horizontal component. Use $V_x = V \cos \theta$

Substitute: $V_x = 12 \cos 30^\circ$

Since $\cos 30^\circ = \frac{\sqrt{3}}{2}$, we have: $V_x = 12 \left(\frac{\sqrt{3}}{2} \right)$

Multiply and divide: $V_x = 6\sqrt{3}$

Simplify: $V_x = 6(1.732) = 10.39$

Hence, the horizontal component vector is 10.



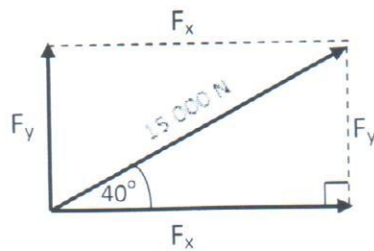
Example 2

A tow truck is pulling a car from a ditch. The tension in the cable is 15 000 N at an angle of 40° to the horizontal.

- Draw a diagram showing the resolution of the force into its rectangular components.
- Determine the magnitudes of the horizontal and vertical components of the force to the nearest whole number.

Solution:

- The tension can be resolved into two rectangular components: vertical component F_y and the horizontal component F_x .



- Magnitude of the horizontal component

Use $V_x = V \cos \theta$

Substitute:

$$F_x = 15\,000 \cos 40^\circ$$

Since $\cos 40^\circ = 0.766$, we have: $F_x = 15\,000(0.76604)$

Multiply

:

$$F_x = 114\,909.6$$

Hence, the magnitude of the horizontal component vector is 114 090 N.

- Magnitude of the vertical component

Use $V_x = V \sin \theta$

Substitute:

$$F_x = 15\,000 \sin 40^\circ$$

Since $\sin 40^\circ = 0.6427$, we have: $F_x = 15\,000(0.6427)$

Multiply

:

$$F_x = 9641.8$$

Hence, the magnitude of the vertical component vector is 9642 N.

Example 3

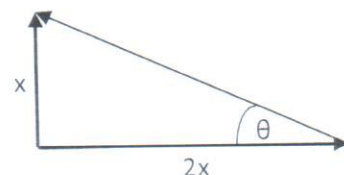
The magnitude of the horizontal component of a force is twice the magnitude of the vertical component. Find the angle of resultant force makes with the larger component.

Solution:

Let the angle be θ .

The vertical component be x

The horizontal component be $2x$





Use the tangent function, since: $\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$

Substitute: $\tan \theta = \frac{x}{2x}$

Cancel x: $\tan \theta = \frac{1}{2} = 0.5$

Find $\arctan 0.5$ or $\tan^{-1}0.5$: $\theta = 26.57^\circ$ or 27°

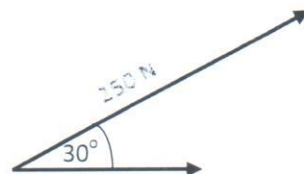
Therefore, the angle the resultant force makes with the horizontal component is 27° .

Example 4

A force of 250 N acts at an angle of 30° to the positive x-axis. Resolve this force into components parallel to the x and y-axes.

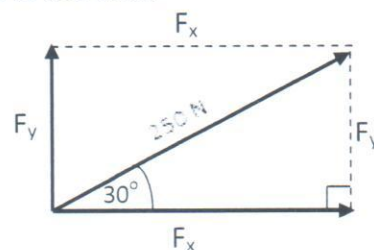
Solution:

First we draw the original vector.



Next we resolve the force into components parallel to the axes.

Since these directions are perpendicular to one another, the components form a right angled triangle with the original force as its hypotenuse.



Now we can calculate the components of the original displacement.

For the magnitude of the vertical component (F_y):

Use $V_x = V \sin \theta$

Substitute: $F_x = 250 \sin 30^\circ$

Since $\sin 30^\circ = 0.5$, we have: $F_x = 250(0.5)$

Multiply: $F_x = 125 \text{ N}$

Hence, the magnitude of the vertical component vector is 125 N.

For the magnitude of the vertical component (F_x):

Use $V_x = V \cos \theta$

Substitute: $F_x = 250 \cos 30^\circ$

Since $\cos 30^\circ = 0.86602$, we have: $F_x = 250(0.86602)$

Multiply: $F_x = 216.5 \text{ N}$

Hence, the magnitude of the horizontal component vector is 216.5 N.

**LEARNING ACTIVITY 12.2.3.5**

20 minutes

Solve the following problems.

1. Find the rectangular components of a vector with magnitude of 16 acting at a point making a 30° angle with the x-axis to the nearest integer.

2. A tow truck is pulling a car from a ditch. The tension in the cable is 12 000 N at an angle of 54° to the horizontal.
 - a) Draw a diagram showing the resolution of the force into its rectangular components.

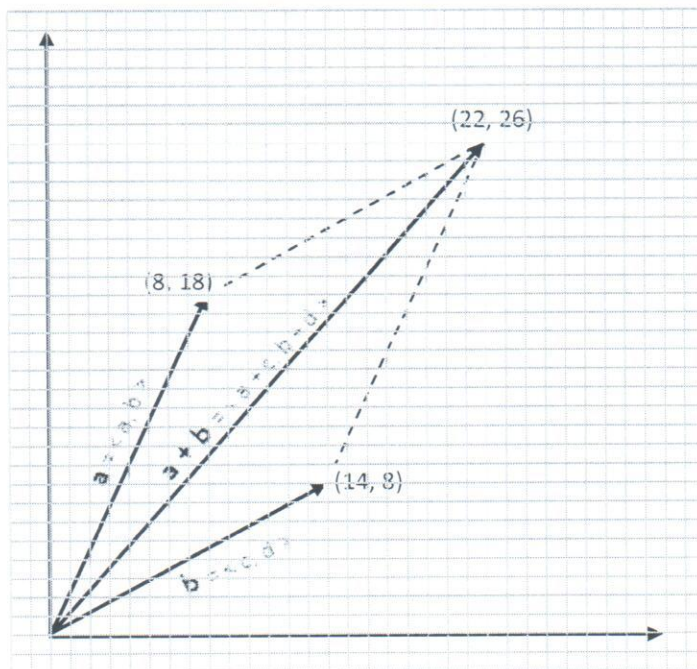
 - b) Determine the magnitudes of the horizontal and vertical components of the force to the nearest whole number.

3. The magnitude of the horizontal component of a force is three the magnitude of the horizontal component. Find the angle of resultant force makes with the smaller component.



12.2.3.6: Algebraic Addition of Vectors

An **algebraic vector** is an ordered pair of numbers. One corresponding to a vector $\mathbf{a}$ in the following figure is represented by the ordered pair (a, b) ; a and b are known as the components of the vector, a being the horizontal component and b being the vertical component.



Algebraic vector addition

Here are few basic facts that will help you in working with algebraic vectors.

- Two vectors with the same components are known as **equal vectors**.
- If both components of a vector are equal to zero, the vector is called a **zero vector**.
- The **magnitude** of a vector, $\mathbf{v} = (a, b)$ is found by doing $|\mathbf{v}| = \sqrt{a^2 + b^2}$. This should come as no surprise, since a and b are the horizontal and vertical components of the vector. Think of the rectangular parallelogram of forces in standard position with the diagonal being the hypotenuse of the right triangle on the right. If you are having difficulty visualizing that, go back and look at the figure on page 112.
- **Vector addition** can be algebraically defined as adding corresponding components of vectors. Look at the figure above. Note how the horizontal components of $\mathbf{a}$ and $\mathbf{b}$ have been added to get the horizontal component of the resultant, $\mathbf{a} + \mathbf{b}$. Then the vertical components were added to get the vertical components of $\mathbf{a} + \mathbf{b}$. you can prove it to yourself by using the numerical coordinates of the two vectors and the resultant. For example

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= (a + c, b + d) \\ (22, 26) &= (8 + 14, 18 + 8) \\ (22, 26) &= (22, 26)\end{aligned}$$



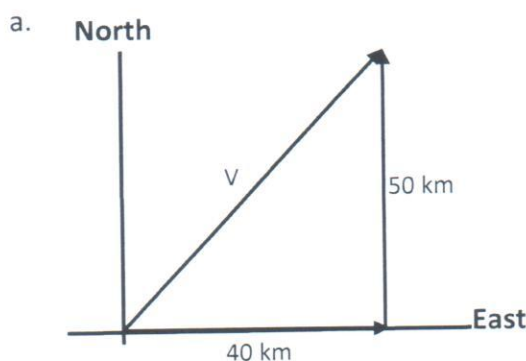
Now look at the following examples.

Example 1

You left your house to visit a friend. You got in your car and drove 40 km east and then get on a highway and went 50 km north.

- Draw a vector from the beginning of your journey, your home, and the end, your friend's house.
- How long is the vector you draw?

Solution:



- Let v be the length or magnitude of the resultant vector you draw.

Using Pythagorean Theorem: $|v| = \sqrt{a^2 + b^2}$
Substitute: $|v| = \sqrt{40^2 + 50^2}$
Square: $|v| = \sqrt{1600 + 2500}$
Combine like terms: $|v| = \sqrt{4100}$
Square root: $|v| = 64.03$
Answer: **64** (to the nearest whole number)

Example 2

Given vector $\mathbf{v} = (-5, 12)$. Find the magnitude of vector $\mathbf{v}$.

Solution:

Remember that a vector has a horizontal and a vertical component that are -5 and 12 respectively. The resultant may be represented as the hypotenuse of a right triangle.

Using Pythagorean Theorem: $|v| = \sqrt{a^2 + b^2}$
Substitute: $|v| = \sqrt{(-5)^2 + 12^2}$
Square: $|v| = \sqrt{25 + 144}$
Combine like terms: $|v| = \sqrt{169}$
Square root: $|v| = 13$
Answer: **13**



Example 3

Given vector $\mathbf{f} = (5, 7)$, vector $\mathbf{g} = (c, d)$. Find the value of d if $\mathbf{f} = \mathbf{g}$.

Solution:

The statement $\mathbf{f} = \mathbf{g}$ means they have the same (or identical) components. $\mathbf{f}$ and $\mathbf{g}$ are equal vectors. This means that they have the same horizontal and vertical components. and the value of $c = 5$.

Therefore, the value of d is 7.

Example 4

Find the magnitude of vector $\mathbf{m} = (-15, -20)$.

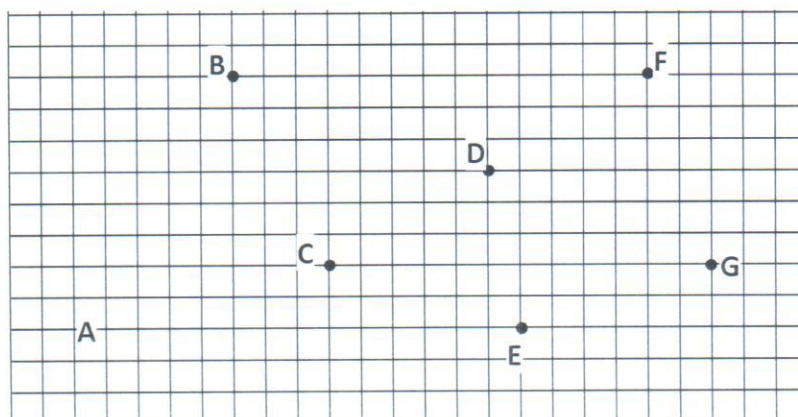
Solution:

Remember that a vector has a horizontal and a vertical component that are -15 and -20 respectively. The resultant may be represented as the hypotenuse of a right triangle.

Using Pythagorean Theorem:	$ \mathbf{m} = \sqrt{a^2 + b^2}$
Substitute:	$ \mathbf{m} = \sqrt{(-15)^2 + 20^2}$
Square:	$ \mathbf{m} = \sqrt{225 + 400}$
Combine like terms:	$ \mathbf{m} = \sqrt{625}$
Square root:	$ \mathbf{m} = 25$
Answer:	25

Example 5

In the grid below, 7 unique locations are provided. Each square on the grid represent 20 m by 20 m area. Rightward on the grid is in the eastward direction and upward on the grid in the northward direction. Use the grid to answer to answer the following question.

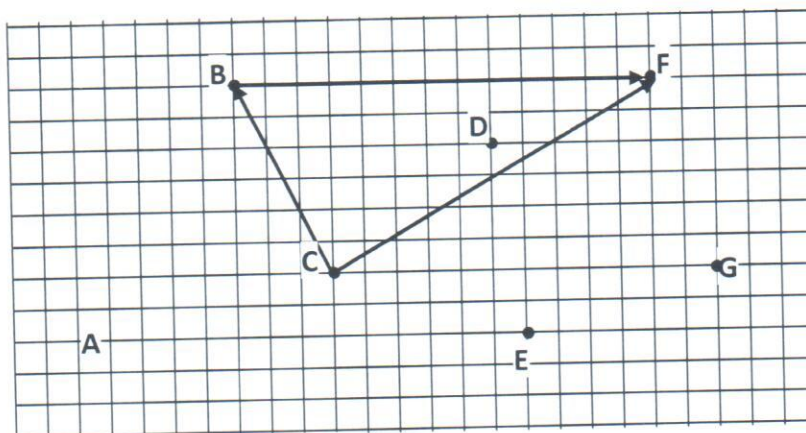




- Suppose that a boy starts at position **C** and walks to position **B** and then to **F**,
- Fill in the table below to indicate the east-west and north-south components of the legs of the walk and the components of the resulting displacement vector.
 - Find the magnitude of the resulting displacement vector.

Solution:

a.



The walk from **C** to **B** is 3 squares to the left (West) and 6 squares up (North). Since each square is 20 metres along each edge, this would be 60 metres West and 120 metres North.

The walk from **B** to **F** is 13 squares to the right (East). This is 260 metres East. There is no north-south component to this leg of the two leg-walk.

The result are shown in the table below.

Vector	East-West Component	North-South Component
C to B	60 metres, West	120 metres, North
B to F	260 metres, East	0 m
Resultant		
C to F	200 metres, East	120 metres, North

- b. The resultant has a horizontal and a vertical component of 200 m and 120 m respectively. To determine the magnitude of the resultant, use the Pythagorean Theorem. $|v| = \sqrt{a^2 + b^2}$

Substitute these to the formula. $|v| = \sqrt{200^2 + 120^2}$

Square: $|v| = \sqrt{40000 + 14400}$

Combine like terms: $|v| = \sqrt{54400}$

Square root: $|v| = 233.23$

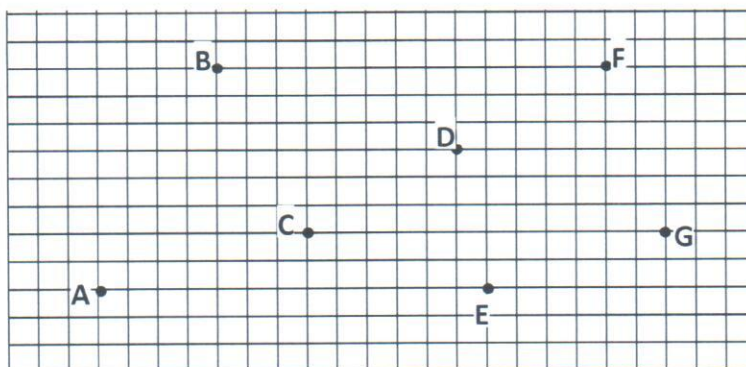
Answer: 233 to the nearest whole metre

**LEARNING ACTIVITY 12.2.3.6**

20 minutes

Solve the following:

1. In the grid below, 7 unique locations are provided. Each square on the grid represent 20 m by 20 m area. Rightward on the grid is in the eastward direction and upward on the grid in the northward direction. Use the grid to answer to answer the following question.



Suppose that a person starts at position **A** and walks to position **E** and then to **G**,

- a. Fill in the table below to indicate the east-west and north-south components of the legs of the walk and the components of the resulting displacement vector.

Vector	East-West Component	North-South Component
A to E		
E to GT		
Resultant		
A to G		

- b. Find the magnitude of the resulting displacement vector.

2. Find the magnitude of vector $\mathbf{h} = (-6, 10)$.



3. Vector $\mathbf{f} = (8, -3)$; vector $\mathbf{g} = (m, n)$. Find the value of n if $\mathbf{f} = \mathbf{g}$.

4. Find the magnitude of vector $\mathbf{m} = (-15, -36)$.

12.2.3.7 Scalar Multiplication

Multiplying a vector by a scalar (an ordinary number) is called **scalar multiplication**. To perform scalar multiplication, you need to multiply the scalar by each component of the vector.

A **scalar** is just a fancy word for a real number. The name arises because a scalar **scales** a vector. That is, it changes the scale of a vector. For example, the real number 2 scales the vector $\mathbf{v}$ by a factor of 2 so that $2\mathbf{v}$ is twice as long as $\mathbf{v}$.

Here's how you multiply the vector using a constant k .

$\mathbf{v} = (x, y)$ by the scalar k .

$$k(\mathbf{v}) = k(x, y)$$

$$k(\mathbf{v}) = (kx, ky)$$

Multiplication by a scalar is commutative. That means it does not matter in which order the factors appear.

$$\text{Vector} \times \text{Scalar} = \text{Scalar} \times \text{Vector}$$

In notation: $k(\mathbf{v}) = \mathbf{v}(k) = (kx, ky)$

For example, you multiply the vector $\mathbf{p} = (3, 5)$ by the scalars 2, -4 , and $\frac{1}{3}$ as follows:

a. $2\mathbf{p} = 2(3, 5) = (6, 10)$

b. $-4\mathbf{p} = -4(3, 5) = (-12, -20)$

c. $\frac{1}{3}\mathbf{p} = \frac{1}{3}(3, 5) = (1, \frac{5}{3})$



Now look at the examples.

Example 1

Multiply $\mathbf{v}(-3, 7)$ by 8.

$$\begin{aligned}\text{Solution:} \quad & \text{Scalar} \times \text{Vector} = 8\mathbf{v} \\ \text{Substitute:} \quad & = 8(-3, 7) \\ \text{Multiply both components by 8:} \quad & = [8(-3), 8(7)] \\ \text{Answer:} \quad & = (-24, 56)\end{aligned}$$

Example 2

If $\mathbf{e} = (c, d)$ and $\mathbf{f} = (g, h)$, find the product of 9 and $\mathbf{e} + \mathbf{f}$.

$$\begin{aligned}\text{Solution:} \quad & \text{Scalar} \times \text{Vector} = 9(\mathbf{e} + \mathbf{f}) \\ \text{Substitute:} \quad & = 9[(c, d) + (g, h)] \\ \text{Multiply both components by 9:} \quad & = 9(c, d) + 9(g, h) \\ \text{Answer:} \quad & = (9c, 9d) + (9g, 9h)\end{aligned}$$

Example 3

If $\mathbf{m} = (7, -8)$ and $\mathbf{n} = (-7, 12)$, find the product of -7 and $\mathbf{m} - \mathbf{n}$.

$$\begin{aligned}\text{Solution:} \quad & \text{Scalar} \times \text{Vector} = -7(\mathbf{m} - \mathbf{n}) \\ \text{Substitute:} \quad & = -7[(7, -8) - (-7, 12)] \\ \text{Multiply both components by 9:} \quad & = -7(7, -8) - 7(-7, 12) \\ \text{Answer:} \quad & = (-49, 56) - (49, -84)\end{aligned}$$

When you multiply a vector by a scalar, the result is a **vector**.

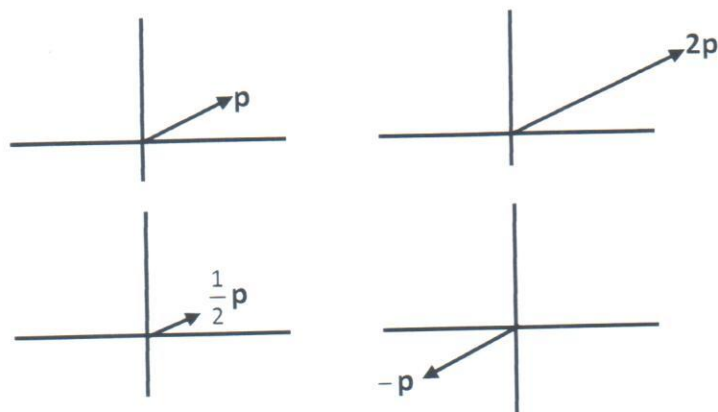
Geometrically speaking, scalar multiplication achieves the following:

- Scalar multiplication by a positive number other than 1 changes the magnitude of the vector but not its direction.
- Scalar multiplication by -1 reverses its direction but doesn't change its magnitude.
- Scalar multiplication by any other negative number both reverses the direction of the vector and changes its magnitude.

Scalar multiplication can change the magnitude of a vector by either increasing it or decreasing it.



- Scalar multiplication by a number greater than 1 or less than -1 **increases** the magnitude of the vector.
- Scalar multiplication by a fraction between -1 and 1 **decreases** the magnitude of the vector.

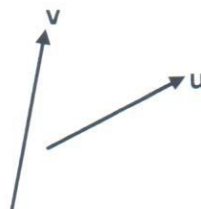


Scalar multiplication of a vector changes its magnitude and/or its direction.

For example, the vector $2\mathbf{p}$ is twice as long as $\mathbf{p}$, the vector $\frac{1}{2}\mathbf{p}$ is half as long as $\mathbf{p}$, and the vector $-\mathbf{p}$ is the same length as $\mathbf{p}$ but extends in the opposite direction from the origin (as shown above).

Dot Products

These are two **vectors**:



They can be **multiplied** using the "**Dot Product**". The dot product is also known as **scalar multiplication**, not to be confused with **multiplication by a scalar**.

The Dot Product is written using a central dot:

$$\mathbf{v} \cdot \mathbf{u}$$

This means the Dot Product of $\mathbf{v}$ and $\mathbf{u}$. If the components of $\mathbf{v}$ are (x_v, y_v) and the components of $\mathbf{u}$ (x_u, y_u) , then $\mathbf{v} \cdot \mathbf{u} = x_u x_v + y_u y_v$ - the sum of the product of the horizontal components and the product of the vertical components.

Example 1

Multiply $\mathbf{v} (-3, 7)$ by $\mathbf{u} (-6, 4)$.

Solution: By formula: $\mathbf{v} \cdot \mathbf{u} = x_u x_v + y_u y_v$

Substitute: $\mathbf{v} \cdot \mathbf{u} = -3 \cdot -6 + 7 \cdot 4$

Multiply: $\mathbf{v} \cdot \mathbf{u} = 18 + 28$

Add: $\mathbf{v} \cdot \mathbf{u} = 46$



Example 2

If $\mathbf{n} = (c, d)$ and $\mathbf{o} = (g, h)$, find the dot product of $\mathbf{n} \cdot \mathbf{o}$.

Solution: By formula: $\mathbf{v} \cdot \mathbf{u} = x_u x_v + y_u y_v$

Substitute: $\mathbf{n} \cdot \mathbf{o} = c \cdot g + d \cdot h$

Multiply: $\mathbf{v} \cdot \mathbf{u} = cg + dh$

And that is as far as we can go.

Example 3

If $\mathbf{d} = (7, -8)$ and $\mathbf{e} = (-7, 12)$, find the dot product of $\mathbf{d} \cdot \mathbf{e}$.

Solution: By formula: $\mathbf{v} \cdot \mathbf{u} = x_u x_v + y_u y_v$

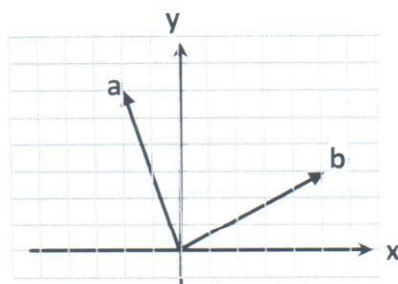
Substitute: $\mathbf{d} \cdot \mathbf{e} = (7)(-7) + (-8)(12)$

Multiply: $\mathbf{d} \cdot \mathbf{e} = -49 + -96$

Add: $\mathbf{d} \cdot \mathbf{e} = -145$

Example 4

What is the value of $\mathbf{a} \cdot \mathbf{b}$ in the figure below?



Solution: $\mathbf{a} = (-2, 6) \Rightarrow x_v = -2, y_v = 6$

$\mathbf{b} = (5, 3) \Rightarrow x_u = 5, y_u = 3$

By formula: $\mathbf{v} \cdot \mathbf{u} = x_u x_v + y_u y_v$

Substitute: $\mathbf{a} \cdot \mathbf{b} = (-2)(5) + (6)(3)$

Multiply: $\mathbf{a} \cdot \mathbf{b} = -10 + 18$

Add: $\mathbf{a} \cdot \mathbf{b} = 8$

Now do the learning activity on the next page.



LEARNING ACTIVITY 12.2.3.7



20 minutes

Solve the following problems.

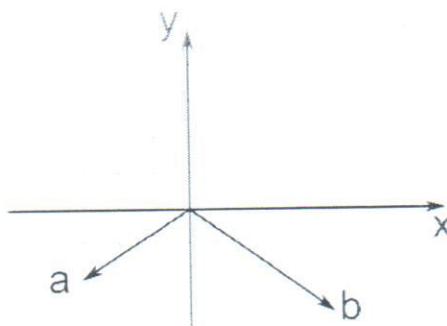
1. If $\mathbf{e} = (c, d)$ and $\mathbf{f} = (g, h)$, find the dot product of -8 and $\mathbf{e} + \mathbf{f}$.

2. If $\mathbf{r} = (6, 9)$ and $\mathbf{s} = (8, -10)$, find the product of -4 and $\mathbf{r} - \mathbf{s}$.

3. Multiply $\mathbf{w} (6, 5)$ by $\mathbf{z} (-5, 6)$.

4. If $\mathbf{m} = (4, -6)$ and $\mathbf{n} = (5, -7)$, find the dot product of $\mathbf{m}$ and $\mathbf{n}$.

5. What is the value of $\mathbf{a} \cdot \mathbf{b}$ in the figure below?

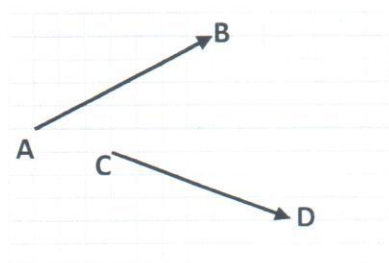




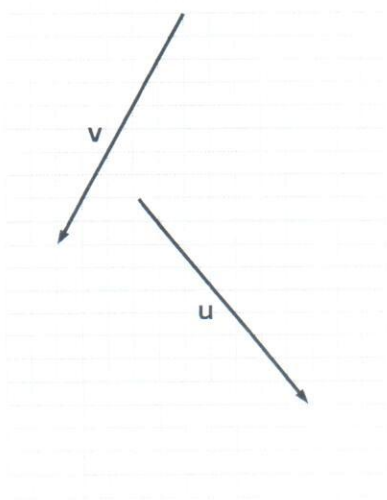
SUMMATIVE TASK 12.2.3

Solve these problems for even more practice applying the skills you learned from this section. The Answer section will direct you to where you need to review.

1. Draw a vector $\mathbf{s}$ to represent a wind blowing south at 100 km per hour for one hour and a vector $\mathbf{p}$ representing an airplane travelling east that has gone 300 miles.
2. Draw a vector representing a weight of 230 kg.
3. Draw the resultant of vectors $\overline{AB}$ and $\overline{CD}$ as shown in the figure on the grid at the right.

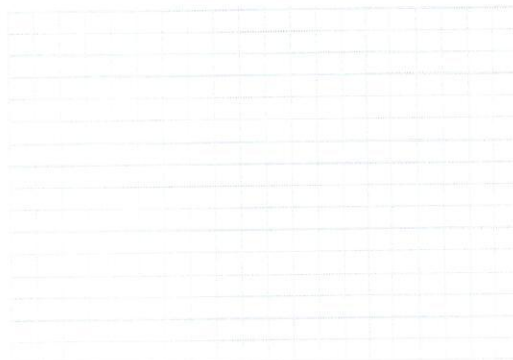
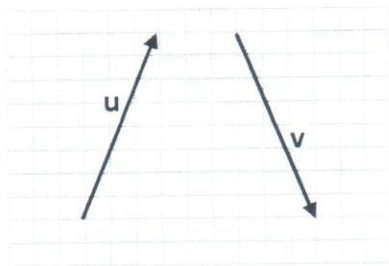


4. Draw the resultant of vector $\mathbf{v}$ and $\mathbf{u}$ as shown in the figure on the grid at the right.

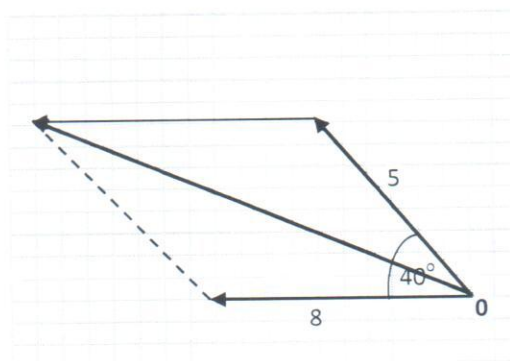




5. Draw the resultant of vectors u and v as shown in the figure below.



6. Solve the parallelogram of forces in the figure below.



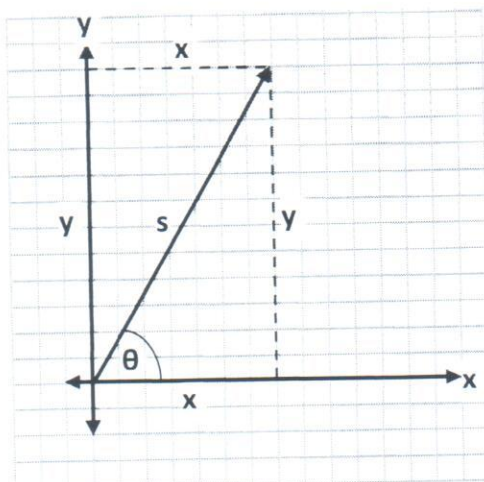
7. A force of 14 kg and a force of 12 kg are acting on an object at an angle of 90 degrees. What are the magnitude and angle of the resultant with respect to the 14 kg force?
8. An airplane is trying to fly due east at 600 km per hour. Because of a wind from the north, the plane's actual path is 10° south of west. Assuming the wind to be blowing at a steady pace, what is its magnitude, to the nearest whole km per hour?



9. A force of 10 kg and a force of 24 kg are acting on an object at an angle of 90 degrees. What are the magnitude and angle of the resultant with respect to the 24 kg force?
10. The endpoints of vector $\overline{AB}$ are at A (-3, -7) and B (2, 5). What are the coordinates of standard vector $\overline{OP}$, if $\overline{OP} = \overline{AB}$?
11. The endpoints of vector $\overline{CD}$ are at C (-6, -9) and D (2, 4). What are the coordinates of standard vector $\overline{OM}$, if $\overline{OM} = \overline{CD}$?
12. The endpoints of vector $\overline{AB}$ are at A (3, 6) and B (13, 14). What are the coordinates of standard vector $\overline{OP}$, if $\overline{OP} = \overline{AB}$?
13. Find the magnitude of vector $\mathbf{h} = (-6, 10)$.
14. Vector $\mathbf{f} = (8, -3)$; vector $\mathbf{g} = (l, m)$. Find the value of m.
15. Find the magnitude of vector $\mathbf{p} = (-15, -36)$.



Refer to the figure below to answer questions 16 – 19.



16. Find the rectangular components of the vector with magnitude of 16 acting at a point making a 30° angle with the x-axis to the nearest integer.
17. The rate and angle with the bank of a boat crossing a stream in kilometres per hour is 15 kph and 70° . What are the cross-stream and current components of the boat's velocity?
18. The vertical component of a force is three times the magnitude of the horizontal component. Find the angle the resultant force makes with the smaller component.
19. Find the rectangular components of the vector with magnitude of 20 acting at a point making a 36° angle with the x-axis to the nearest integer.



-
20. Multiply $\mathbf{v}$ $(-5, 8)$ by 7.
21. Vectors $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = (2, -3)$ and $\mathbf{B} = (-4, 2)$. Find the dot product $\mathbf{A} \cdot \mathbf{B}$ of the two vectors.
22. Vector $\mathbf{A}$ is given by $\mathbf{A} = (4, -3)$. Find $|\mathbf{A}|$.
23. If $\mathbf{m} = (7, -8)$ and $\mathbf{n} = (6, -9)$, find the product of -7 and $\mathbf{m} - \mathbf{n}$.

Refer to the information to answer questions 24 and 25.

John was rowing directly across a river at the rate of 4 kilometres per hour. The current was flowing at a rate of 3 kilometres per hour.

24. Draw each vector to represent the path of the boat.
25. Determine the magnitude of the resultant velocity of the boat by measuring the vector.



UNIT SUMMARY

This summary outlines the important key ideas and concepts to be remembered.

12.2.1 Trigonometry

- An **angle** is determined by rotating a ray about its endpoint.
- The **initial side** of an angle is the ray where the measurement of an angle starts.
- The **terminal side** of an angle is the ray where the measurement of an angle stops.
- The **measure of an angle** is determined by the amount of rotation from the initial side to the terminal side.
- The **radian** is the standard unit of angular measure. It is the measure of an angle θ that, when drawn as a central angle subtends an arc whose length equals the length of the radius of the circle.
- To convert degrees to radians measure, multiply $\frac{\pi}{180^\circ}$.
- To convert radians to degrees measure, multiply $\frac{180^\circ}{\pi}$.
- The **length of an arc** is the length of the portion of the circumference of a circle.
- The formula for finding the length of an arc: $C_A = \frac{\theta}{360^\circ} \times 2\pi r$
- The **sector of a circle** is a pie-shaped region bounded by two radii and an arc lying between the radii.
- The formula for finding the sector of a circle: $A = \frac{n}{360^\circ} \pi r^2$
- The segment of a circle is the region bounded by a chord and the arc subtended by the chord.
- The formula for finding the area of a segment of a circle: $A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}$
- The trigonometric functions:

$\sin \theta = y$	$\csc \theta = \frac{1}{y} \quad (y \neq 0)$
$\cos \theta = x$	$\sec A = \frac{1}{x} \quad (x \neq 0)$
$\tan \theta = \frac{y}{x} \quad (x \neq 0)$	$\cot A = \frac{x}{y} \quad (y \neq 0)$
- **Reference angle** is the acute angle θ formed by the terminal side of θ and the horizontal axis.
- The formulas in finding the reference angles for Quadrants II, III and IV.

Quadrant II	Quadrant III	Quadrant IV
$\theta' = \pi - \theta$ (radians)	$\theta' = \pi - \theta$ (radians)	$\theta' = 2\pi - \theta$ (radians)
$\theta' = 180^\circ - \theta$ (degrees)	$\theta' = 180^\circ - \theta$ (degrees)	$\theta' = 360^\circ - \theta$ (degrees)
- To evaluate trigonometric functions of any angle:
 - Determine the function value of the associated reference angle θ .
 - Depending on the appropriate quadrant in which θ lies, affix the appropriate sign to the function.



- The **amplitude** of a function is one half the distance between its maximum and its minimum values.
- The **period** of a function is the horizontal length of one repeating pattern or interval.
- A function is said to be **periodic** when the values of the function regularly repeat themselves.
- A horizontal shift of the trigonometric function is when the entire graph shifts left or right along the x-axis.
- The vertical shift of the trigonometric functions occurs when the entire graph shifts up or down along the y-axis.
- The Sine Rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- The Cosine Rule: $a^2 = b^2 + c^2 - 2bc \cos A$
 $b^2 = a^2 + c^2 - 2ac \cos B$
 $c^2 = a^2 + b^2 - 2ab \cos C$

12.2.2 Trigonometric Identities and Equations

- Trigonometric equations are equations involving trigonometric functions of unknown angles.
- Trigonometric identities are equations that are true for all values of the variable in a specified domain.
- Reciprocal identities are the simplest and most basic trigonometric identities. They are those involving the reciprocals of the trigonometric functions.
- Ratio identities are also referred to as the quotient identities.

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

- The Pythagorean Identities:
 $\sin^2 \theta + \cos^2 \theta = 1$ $\tan^2 \theta + 1 = \sec^2 \theta$ $\cot^2 \theta + 1 = \csc^2 \theta$
- Addition and subtraction identities are identities that use two variables. They are simply known as trigonometric addition identities. (see p77)
- To prove an identity, use a logical steps showing that the one side of the equation ends up with the other side of the equation. To do this start with either the LHS or the RHS and choose the side that looks more readily able to be simplified. Re-express the terms using identities so that it takes the form of the other side.
- To solve trigonometric equations, we use the rules of algebra to isolate the trigonometric function on one side of the equal sign.

12.2.3 Vectors

- A **vector** is a quantity that has both magnitude and direction.
- A **scalar** is a quantity that has magnitude but no direction.
- A vector with its initial point at the origin is called **standard vector** or position vector.
- Any vector in the coordinate plane that does not have a starting point is called a **free vector**.
- **Resolution of vector** is the process of splitting a vector into various parts which are called components of vector.
- The x-components is called the **horizontal components**.



The magnitude of the horizontal components is found by using the formula:

$$V_x = V \cos \theta$$

- The y-components is called the **vertical components**.

The magnitude of the vertical components is found by using the formula:

$$V_y = V \sin \theta$$

- **Addition of vectors or the Tip/Tail Rule** (vector triangle)
To add vectors, place the tip of one to the tail of an equivalent of the other which resulted to what we called **resultant vector**.
- **Parallelogram Law of forces or Vector parallelogram:** A parallelogram having two adjacent sides that represent two force vectors and an included diagonal that represents the vector sum. It is used to determine the resulting force when two vectors act on an object.
- **Algebraic addition of vectors:** To add vectors algebraically:
 - Find the x- and y-components of each vector
 - Add the x and y components of each vector.
 - Draw a resultant vector
 - Determine the magnitude of the resultant with the Pythagorean Theorem.
 - Calculate the angle of the displacement using inverse tangent.
- If both the initial point and the terminal point of a vector are at the origin it is called **zero vector**. This means that both components of a vector are equals to zero.
- Scalar Multiplication is multiplying a vector by a scalar.
- When we multiply a vector by a scalar, the result is a vector.
- Scalar Multiplication is commutative. That means it does not matter in which order the factors appear.
$$\text{Vector} \times \text{Scalar} = \text{scalar} \times \text{Vector}$$
$$k(\mathbf{v}) = \mathbf{v}(k) = (kx, ky)$$
- Two vectors can be multiplied using the **Dot product**, which is written as:

$$\mathbf{v} \cdot \mathbf{u}$$

If the components of $\mathbf{v}$ are (x_v, y_v) and the components of $\mathbf{u}$ are (x_u, y_u)

Then

$$\mathbf{v} \cdot \mathbf{u} = x_u x_v + y_u y_v$$

REVISE SECTIONS 12.2.1 TO 12.2.3 THEN DO MODULE ASSESSMENT 2

**ANSWERS TO LEARNING ACTIVITIES 12.2.1.TO 12.2.3**

Learning Activity 12.3.1.1

- | | | | | | |
|----|-------|------------------|----|----|----------------|
| 1. | a. | $\frac{3\pi}{2}$ | 2. | a. | 90° |
| | b. | $\frac{7\pi}{6}$ | | b. | 315° |
| | c. | $\frac{3\pi}{4}$ | | c. | 114.64° |
| | d. | $\frac{5\pi}{3}$ | | d. | 150° |
| | a. e. | $\frac{\pi}{5}$ | | e. | 140° |
-

Learning Activity 12.3.1.2

- | | | | | | | | |
|----|------|----|---------|----|-------|----|---------|
| 1. | 9 cm | 2. | 6.28 cm | 3. | 14 cm | 4. | 13.6 cm |
|----|------|----|---------|----|-------|----|---------|
-

Learning Activity 12.3.1.3

- 61.4 cm²
 - 29.6 cm²
 - 15.7 sq.units
 - 3 sq. metres
 - 12.4 sq. units
 - $\frac{15\pi}{2}$
-

Learning Activity 12.3.1.4.1

- Quadrant II
 - Positive** (100° is in the second quadrant and sine is positive).
 - Negative** (188° is in the third quadrant and cosine is negative there).
 - Positive** (-15° is in the fourth quadrant. Since negative angles are measured clockwise. Since cosine is positive in the fourth quadrant, secant will also be positive.)
 - Positive** (225° is in the third quadrant and tangent is positive there).
-



3. a. 120° lies in 2nd Quadrant. So, $\theta' = 180^\circ - \theta = 180^\circ - 120^\circ = 60^\circ$
b. 210° is in the 3rd Quadrant. So, $\theta' = \theta - 180^\circ = 210^\circ - 180^\circ = 30^\circ$
c. 330° is in the 4th Quadrant. So, $\theta' = 360^\circ - \theta = 360^\circ - 330^\circ = 30^\circ$
d. $2.5 = 143.31^\circ$ which lies between 90° and 180° . It follows that it is in 2nd Quadrant and its reference is $\theta' = 180^\circ - \theta = 180^\circ - 143.31^\circ = 36.69^\circ$

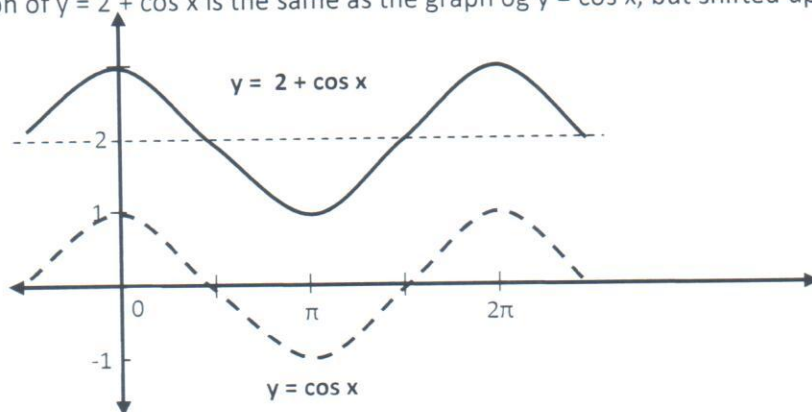
Learning Activity 12.3.1.4.2

1. a. $-\frac{\sqrt{2}}{2}$ b. $\sqrt{3}$ c. $\frac{1}{2}$ d. -2
2. a. $-\frac{1}{2}$ b. $-\sqrt{3}$ c. $\frac{\sqrt{2}}{2}$
3. a. 2 b. 0

Learning Activity 12.3.1.4.3

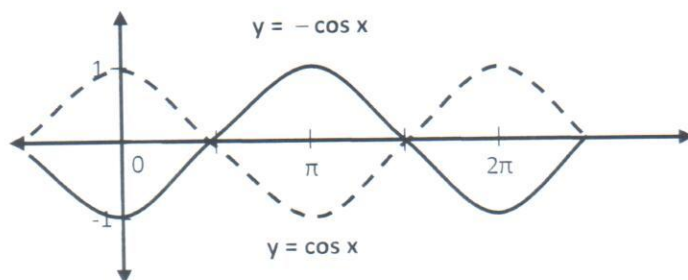
A. 1.

The graph of $y = 2 + \cos x$ is the same as the graph of $y = \cos x$, but shifted up 2 units.



2.

The graph of $y = -\cos x$ is the reflection of the graph of $y = \cos x$.





B. 1. $y = 2 \cos 3x$

Amplitude = $|2| = 2$

Period = $\frac{2\pi}{3}$

Phase shift = 0

2. $y = -5 \cos \frac{1}{3}x$

Amplitude = $|-5| = 5$

Period = $\frac{2\pi}{\frac{1}{3}} = 2\pi(3) = 6$

Phase shift = 0

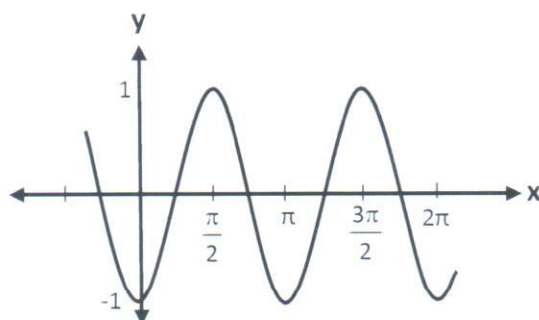
3. $y = -2 \sin 4(x - 1.3)$

Amplitude = $|-2| = 2$

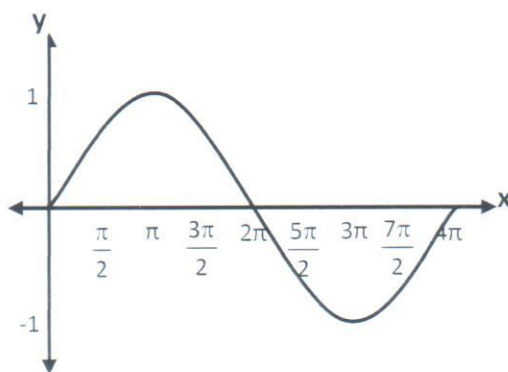
Period = $\frac{2\pi}{4} = \frac{\pi}{2}$

Phase shift = 1.3 (shift 1.3 units to the right)

C. 1. $y = -\cos 2x$



2. $y = \sin\left(\frac{1}{2}x\right)$



The graph reaches its maximum in this interval at π .

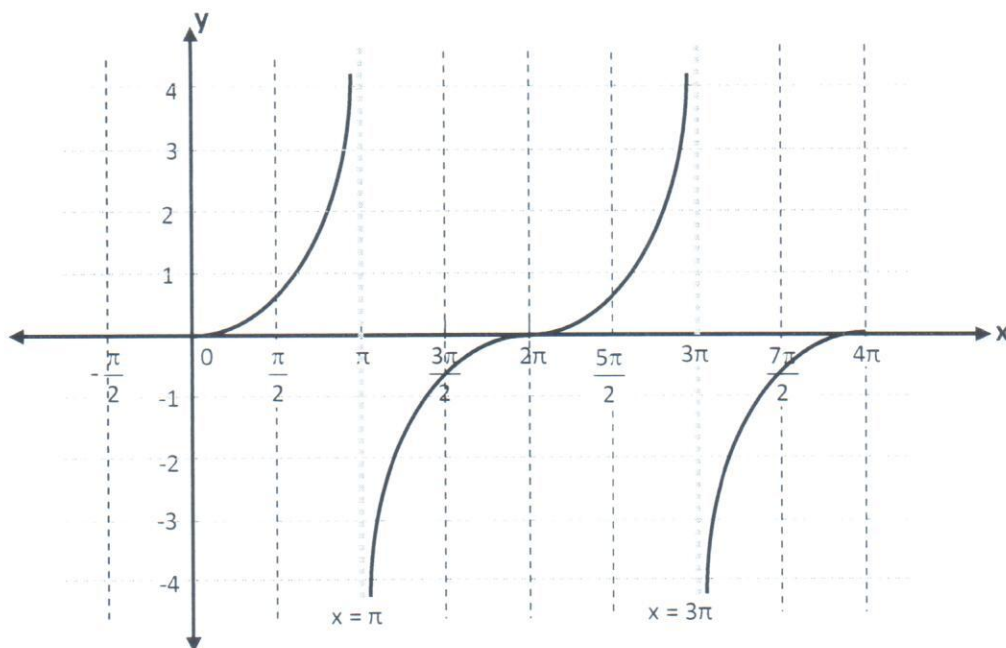


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-



3. Graph of $y = 2 \tan \frac{x}{2}$ in the domain $\{0 \leq x \leq 4\pi\}$.



Learning Activity 12.2.1.6.1

- 58.99°
- a. 3.44 cm b. 69° c. 5.75 cm
- a. 11.2 cm b. 44°

Learning Activity 12.2.1.4.6.2

- a. 21 cm b. 31 cm c. 43 cm d. 6.8 cm
- a. 60° b. 17° c. 105° d. 58.8°

Summative Task 12.2.1

- a. $\frac{\pi}{5}$ b. $-\frac{\pi}{4}$ c. π d. $\frac{217\pi}{180}$
- a. 270° b. 345° c. -210° d. 570°



3. a. $\frac{4}{3}$ b. $\frac{4}{3}$ c. $\frac{4}{5}$ d. $\frac{12}{25}$

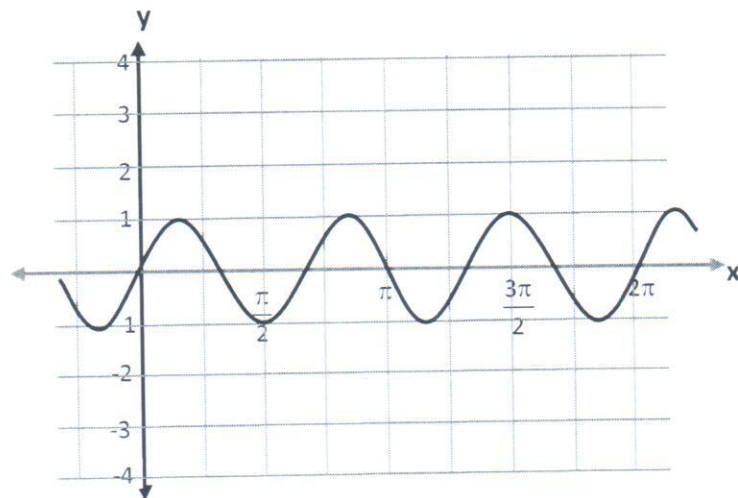
4. a. $-\sqrt{3}$ b. $-\sqrt{2}$ c. $\frac{\sqrt{3}}{2}$
d. 1 e. -1 e. $-\frac{\sqrt{3}}{3}$

5. a. 52.4 cm^2 b. $147\pi \text{ cm}^2$

6. a. $y = \sin 3x$

Amplitude = 1

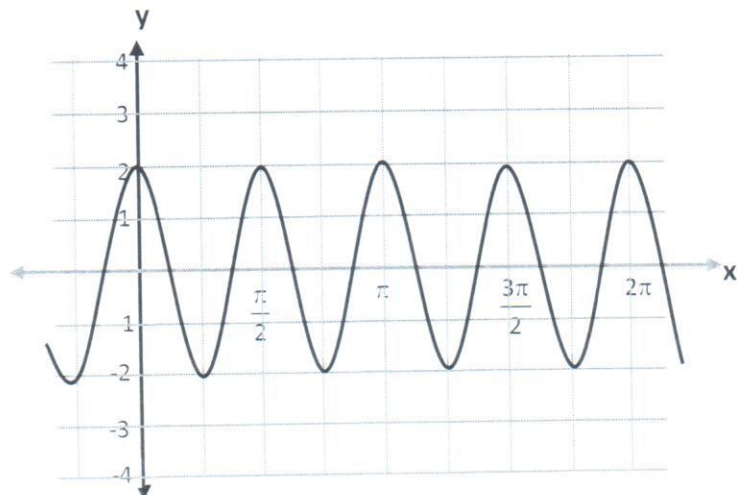
Period = $\frac{2\pi}{3}$



b. $y = 2 \cos 4x$

Amplitude = 2

Period = $\frac{\pi}{2}$



**Learning Activity 12.2.2.1**

1. $\frac{5\sqrt{11}}{11}$

2. $\sec \theta$

3. 7

4. $\cos \theta = \pm \frac{4}{5}, \tan \theta = \pm \frac{3}{4}, \cot \theta = \pm \frac{4}{3}, \sec \theta = \pm \frac{5}{4}, \csc \theta = \frac{5}{3}$

Learning Activity 12.2.2.2

a. $\sin \theta \cot \theta = \cancel{\sin \theta} \cdot \frac{\cos \theta}{\cancel{\sin \theta}}$
 $= \cos \theta$

b. $\sec \theta \cot \theta = \frac{1}{\cancel{\cos \theta}} \cdot \frac{\cancel{\cos \theta}}{\sin \theta}$
 $= \frac{1}{\sin \theta}$
 $= \csc \theta$

c. $(1 + \tan^2 \theta) \sin^2 \theta = (\sec^2 \theta) \sin^2 \theta$
 $= \frac{1}{\cos^2 \theta} (\sin^2 \theta)$
 $= \frac{\sin^2 \theta}{\cos^2 \theta}$
 $= \tan^2 \theta$

d. $\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$
 $= \sec^2 \theta$

e. $\frac{\csc^2 \theta + \cot^2 \theta}{\sec^2 \theta} = \frac{1}{\sec^2 \theta}$
 $= \cos^2 \theta$

Learning Activity 12.2.2.3

1. $\frac{\sqrt{3}}{2}$

2. $\frac{\sqrt{3} \sin x - \cos x}{2}$

3. $\frac{1}{2}$



4. $\sqrt{2} \cos 2x$

5. -13

6. $\frac{7}{6}$

Learning Activity 12.2.2.4

1. $-\frac{119}{169}$

2. $\frac{24}{25}$

3. $-\frac{24}{7}$

Learning Activity 12.2.2.5

1. $-\frac{\sqrt{5}}{4}$

2. $\sqrt{2} - 1$

3. $\sqrt{7}$

4. $\frac{\sqrt{14}}{3}$

Learning Activity 12.2.2.6

1. a. $\theta = 135^\circ, 315^\circ$

b. $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

c. No solution

d. $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

2. $\frac{\pi}{3}, \frac{2\pi}{3}$

3. $0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$

Summative Task 12.2.2

1.

$$\text{Answers: } \sin \theta = \pm \frac{3}{5}, \tan \theta = \pm \frac{3}{4}, \cot \theta = \pm \frac{4}{3}, \sec \theta = -\frac{5}{4}, \csc \theta = \pm \frac{5}{3}$$

2. a. $\cot \theta + \csc \theta$

$$\text{Answer: } \frac{\cos \theta + 1}{\sin \theta}$$

b. $\sin \theta \cot \theta + \csc \theta$

$$\text{Answer: } \cos \theta + \frac{1}{\sin \theta}$$



3. a. $1 - 2 \sin^2 \theta$

c. $\frac{1 + \sin^2 \theta}{1 - \sin^2 \theta}$

4. a. $\cos \theta \tan \theta = \cancel{\cos \theta} \cdot \frac{\sin \theta}{\cancel{\cos \theta}}$
 $= \sin \theta$

c. $\cot \theta \cos \theta = \frac{\cos \theta}{\sin \theta} \cdot \cos \theta$
 $= \frac{\cos^2 \theta}{\sin \theta}$
 $= \frac{1 - \sin^2 \theta}{\sin \theta}$
 $= \frac{1}{\sin \theta} - \sin \theta$
 $= \csc \theta - \sin \theta$

b. $\frac{1 - 2 \sin^2 \theta}{\pm \sin^2 \theta \sqrt{1 - \sin^2 \theta}}$

d. $\frac{1}{\sin \theta (1 - \sin^2 \theta)}$

b. $\sin \theta \sec \theta = \sin \theta \cdot \frac{1}{\cos \theta}$
 $= \frac{\sin \theta}{\cos \theta}$
 $= \tan \theta$

d. $\frac{\tan \theta - 1}{\tan \theta + 1} = \frac{1 - \frac{1}{\tan \theta}}{1 + \frac{1}{\tan \theta}}$
 $= \frac{1 - \cot \theta}{1 + \cot \theta}$

5. a. $\sin(60^\circ + 30^\circ) = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$
 $= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2}$
 $= \frac{3}{4} + \frac{1}{4} = \frac{4}{4} = 1$

b. $\sin(45^\circ + 45^\circ) = \sin 45^\circ \cos 45^\circ + \cos 45^\circ \sin 45^\circ$
 $= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}$
 $= \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$

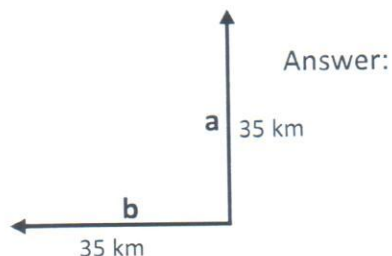
2. $\cos 2\theta = 1 - 2 \sin^2 \theta = 1 - 2 \left(\frac{3}{4}\right)^2 = 1 - 2 \left(\frac{9}{16}\right)$
 $= 1 - \frac{9}{8}$
 $= -\frac{1}{8}$



7. a. $\frac{\sqrt{6} - \sqrt{2}}{4}$ b. $2 + \sqrt{3}$
c. -1 d. -1

Learning activity 12.2.3.1

1.



2



3. The difference between scalar and vector quantities is that the former only provides numerical data or magnitude, while the latter also provides direction in addition to magnitude. Examples of scalar measurements include time, temperature, speed and mass, whereas examples of vectors consist of velocity, acceleration and force.

Learning activity 12.2.3.2

1.

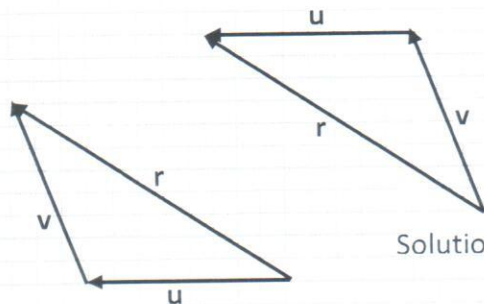


Solution 1



Solution 2

2.

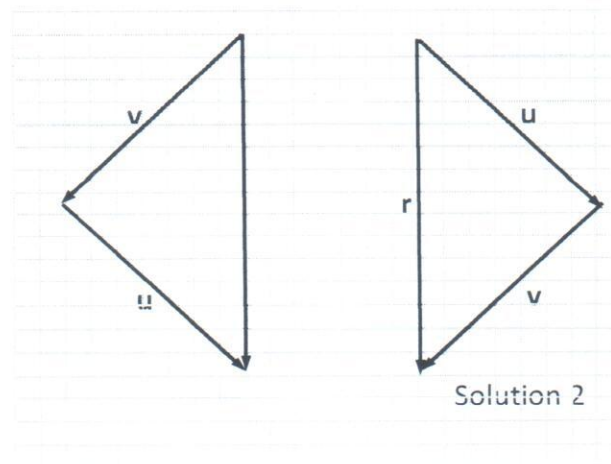


Solution 1

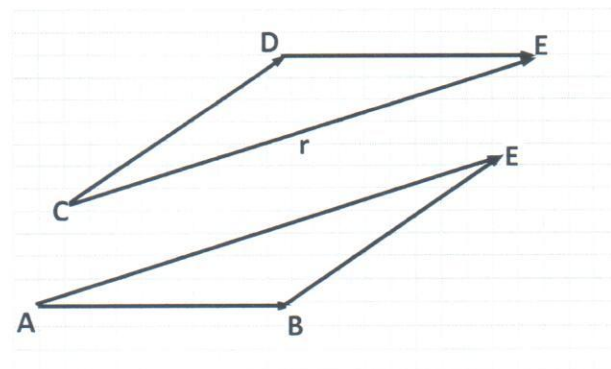
Solution 2



3.



4.

**Learning activity 12.2.3.3**

1. about 66.8 kg at 18.5°
2. The resultant is 42 g in magnitude at 21.8° angle
3. 15 kg at 37°
4. 186.4 kph

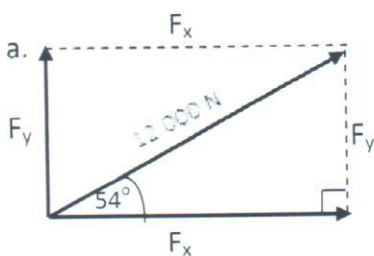
Learning activity 12.2.3.4

1. (10, 12)
2. (9, 11)
3. (10, 12)
4. (8, 13)

**Learning activity 12.2.3.5**

1. $x = 14, y = 8$

2.



b. $F_y = 9.708 \text{ N}; F_x = 7053 \text{ N}$

3. 72°

Learning activity 12.2.3.6

1. a.

Vector	East-West Component	North-South Component
A to E	240 m, East	0 m
E to GT	120 m, East	40 m, North
Resultant		
A to G	360 m, East	40 m, North

b. $v = 362 \text{ metres}$

2. $2\sqrt{34}$

3. -3

4. 39

Learning activity 12.2.3.7

1. $-8(\mathbf{e} + \mathbf{f}) = (-8\mathbf{c}, -8\mathbf{d}) + (-8\mathbf{g}, -8\mathbf{h})$

2. $-4(\mathbf{r} - \mathbf{s}) = (-24, -36) + (-32, 40)$

3. $\mathbf{w} \cdot \mathbf{z} = 0$

4. $\mathbf{m} \cdot \mathbf{n} = 62$

5. $\mathbf{a} \cdot \mathbf{b} = -6$

Summative task 12.2.3

1.

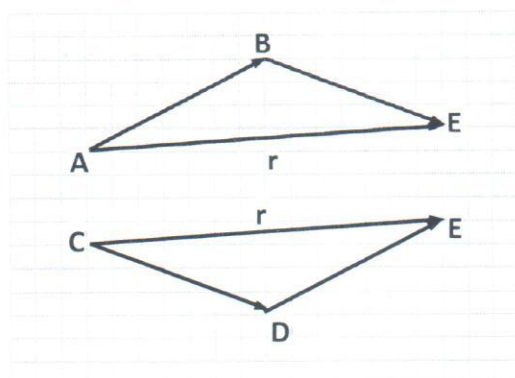


2.

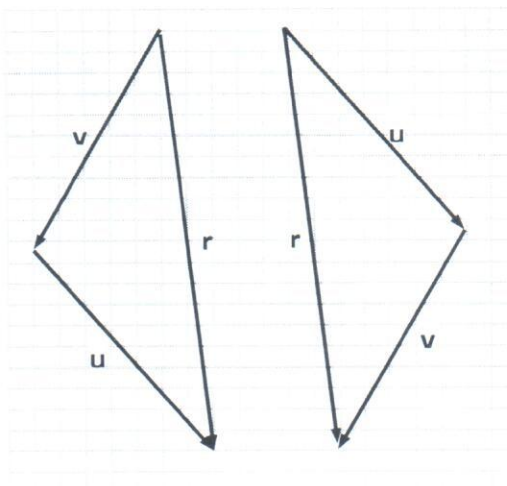




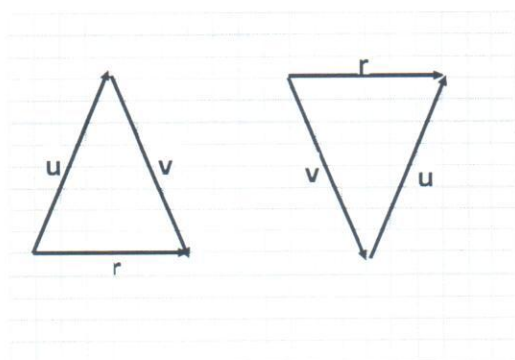
3.



4.

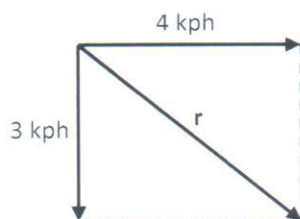


5.



6. 12.3 kg at 15.14°
7. 18.44 kg at 40.60°
8. 105.6 kph
9. 26 kg at 23°
10. (5, 12)
11. (8, 13)
12. (10, 8)
13. $|h| = 2\sqrt{34}$
14. -3
15. 39

16. $x = 14; y = 8$
17. Current is about 5 kph, cross stream rate is about 14 kph
18. 72°
19. $x = 16; y = 12$
20. $7v = (-35, 56)$
21. $\mathbf{A} \cdot \mathbf{B} = -14$
22. $|\mathbf{A}| = 5$
23. $-7(\mathbf{m} - \mathbf{n}) = (-49, 56) - (-42, 63)$
- 24.



25. 5 kph

END OF UNIT MODULE 12.2



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SUBJECT AND GRADE TO STUDY

Grade Levels	Subjects
Grades 7 and 8	1. English
	2. Mathematics
	3. Science
	4. Social Science
	5. Making a Living
	6. Personal Development
	7. English
Grades 9 and 10	1. English
	2. Formal Mathematics
	3. Practical Mathematics
	4. Science
	5. Social Science
	6. Commerce
	7. Design and Technology- Computing
	8. Personal Development
Grades 11 and 12	1. English
	• Applied English
	• Language and Literature
	2. Mathematics
	• Mathematics A
	• Mathematics B
	3. Science
	4. Social Science
	• History
	• Geography
	• Economics
	5. Business Studies
	6. Personal Development
	7. ICT

REMEMBER:

- For Grades 7 and 8, you are required to do all six (6) courses.
- For Grades 9 and 10, you must study English, Mathematics, Science, Personal Development, Social Science and Commerce. Design and Technology-Computing is optional.
- For Grades 11 and 12, you are required to complete seven (7) out of thirteen (13) courses to be certified. Your Provincial Coordinator or Supervisor will give you more information regarding each subject.

Certificate in Matriculation

CORE COURSES

Basic English
English 1
English 2
Basic Maths
Maths 1

Maths 2

History of Science & Technology

OPTIONAL COURSES

Science Streams: Biology

Chemistry, Physics and Social Science Streams:

Geography, Introduction to Economics and Asia and the Modern World

REMEMBER:

You must successfully complete 8 courses; 5 compulsory and 3 optional